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MATHEMATICS magazine

Bird

Mathematics Magazine

Formerly *National Mathematics Magazine* founded by S. T. SANDERS

The overall objective of the Mathematics Magazine is to portray the interests of research mathematicians, teachers of mathematics and others interested in the subject.

Expository papers on modern fields of research will be emphasized.

Immediate publication is made possible by two devices:

First, research papers are published without the delay of refereeing, and then discussed in succeeding issues (see "Current Papers and Books").

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Because of the wide variety of papers to be published, it is highly desirable that each introduction should state what the paper accomplishes, and that each paper should be accompanied by a general statement of what mathematical training is required to read it.

All manuscripts must be typed on 8½" x 11" paper, with 1" margins and double-spaced. Duplicate copies should be sent to us.

For further information concerning the objectives and policies of the Mathematics Magazine see the headings of the various departments and inside of back cover.

Note: The last issues of the *National Mathematical Magazine* were Vol. XX, Numbers 1 and 2, October and November, 1945. Since this is a new series, we are starting this September-October issue as Vol. XXI, No. 1. As a further convenience for librarians and reference seekers, colors of succeeding volumes will be varied.

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Marion E. Stark explains what should be sent to this Department, and supplies interesting samples.

Our Contributors

Brief notes concerning the authors of our papers, designed to increase interest in these papers and to further acquaintance among those interested in mathematics.

Among the papers to appear in early issues of the MATHEMATICS MAGAZINE are the following:

"Analogues for Arithmetical Functions of Elementary Transcendental Functions," bp E. T. Bell; "The Operational Calculus," by F. D. Murnaghan; "On a Property of the Laplacian of a Function," by J. Kampe Feriet; "Functional Analysis and Topolo0ical Groug Spaces," by A. D. Michal: "Concerning Two New Chapters in the Theory of Probability," by Maurice Frechet; "The Generalized Weierstrass Approximation Theorem," by M. H. Stone.

Everyone will be interested in the discussions of the following topics, coming in the next issue, "Opportunities for Mathematically Trained College Students" by I. S. SOKOLNIKOFF and "The Use of Mathematicians in the Airplane Industry" by WILLIAM BOLLAY, North American Aviation, Inc.

This magazine is very much interested in the synthesis of pure and applied mathematics. Accordingly engineers and physicists as well as mathematicians are invited to send us authoritative expository mathematical articles or topics for which they desire us to secure authors.

Orthogonal Polynomials and Polynomials in General

by J. A. Shohat

I. *Role of Polynomials in Analysis and its Applications.* Polynomials i. e. functions of the type (1) $y = a_0 + a_1 x + \cdots + a_n x^n$, n a positive integer or zero, are familiar to us from our high school days. They serve us early as a working bench for mastering the fundamental algebraic operations and a little later, for $n = 2$, we meet them in the study of quadratic equations and their graphic solutions. In College Algebra we deal extensively with (1) by learning first how to differentiate and integrate "powers," and later, at more advanced stages, we again meet (1) when dealing with Maclaurin's Formula and Power Series. In Applied Mathematics the importance of polynomials cannot be overestimated. When the engineer, the physicist, the statistician, the physical chemist try to represent analytically results of experiments or observations, what we call "empirical functions", he first resorts to polynomials, using expressions of the type

$$(2) \qquad a + bx,\ a + bx + cx^2,\ a + bx + cx^2 + dx^3, \cdots .$$

Our "Tables", mathematical, physical, and so on, are mostly computed in this manner.

One is inclined to say that in all such cases we use polynomials only because they are the simplest type of functions. However, simplification is often poor justification. Fortunately, we have a strong foundation upon which we can build the use of polynomials in Pure and Applied Mathematics. First, when dealing with analytic functions, that is, with functions representable by power series, we naturally obtain their polynomial representation by taking a certain number of initial terms in the corresponding Maclaurin (or Taylor) series. But the class of analytic functions, however important, is very limited, is but a tiny island in the vast universe of functions. Moreover, in applications we seldom expect our functions to be analytic. Here comes to our rescue the celebrated Weierstrass' Approximation Theorem, given in 1885 by the great German mathematician Weierstrass (1815-1897), the father of modern rigour in Analysis. We state it as follows:

Let $f(x)$ be an arbitrary continuous function defined on a finite interval (a,b). We can always approximate $f(x)$ as closely as we please on the whole interval (a,b) by a polynomial of sufficiently high degree.

Roughly speaking, we may say that a continuous function over a finite interval may be treated very much like a polynomial. The theorem is far reaching. Indeed, it requires only the continuity of $f(x)$, and we know how complex, how "pathological" may be the behaviour of a continuous function. It may have no derivative at any point of its interval of definition, it may have therein infinitely many maxima and minima, so that the corresponding curve will nowhere have a tangent or will have infinitely many "teeth" in the interval and thus will not be "plottable". And yet, in spite of all this complexity, Weierstrass' Theorem allows us to approximate our function as close as we please *over the whole interval* by such a simple analytical expression as a polynomial, or, in geometrical language, we can approximate our curve indefinitely close by a parabola of a certain order. No wonder this startling theorem is considered the foundation of the modern Theory of Functions of a Real Variable.

Thus, use of polynomials in various applications of Analysis is justified, and their study becomes one of its most interesting and important chapters.

In what follows we attempt to exhibit some results of this study*, mainly without proofs, due to space limitation.

2. *Orthogonal polynomials.* The following problem is typical in the theory of polynomials, and its importance is manifest.

Suppose the polynomial $P(x)$, of given degree n, does not exceed numerically a given constant on a given interval. We wish to estimate $|P'(x)|$, the highest coefficient of $P(x)$, and so on.

The standard form (1) of a polynomial, to which we are so used in Algebra, while of importance in the Theory of Equations, proves inadequate in our discussion. We need here a new representation, with new "building blocks" replacing the familiar powers $x^0, x^1, x^2, \cdots$. We choose for such building blocks the so-called "orthogonal polynomials" (OP),† introduced in their full generality by the Russian mathematician Tchebycheff (1821-1894).

2.1. *The Fundamental Existence Theorem for OP. Let $p(x)$ be defined on a given interval (a,b) finite or infinite, with the following properties:* (i) $p(x) \geqq O$; (ii) *all "moments"*

$$\alpha_n = \int_a^b p(x)x^n dx, \quad n = 0,1,2 \cdots,$$

*We confine our exposition to real-valued functions of one real variable.

†For a more detailed discussion of *OP* the reader is referred to Carus monograph by D. Jackson: Fourier Series and Orthogonal Polynomials.

exist, $\alpha_0>0$. *There exists a uniquely determined, up to constant factors, infinite set of polynomials* $Q_0(x), Q_1(x), Q_2(x), \cdots$, *of degree respectively* $0,1,2,\cdots$, *satisfying the "orthogonality relation."*

$$\int_a^b p(x)Q_m(x)Q_n(x)dx=0, \quad m,n=0,1,2,\cdots; \quad m\neq n. \tag{3}$$

Proof. It is evident that in (3) we can assume $m<n$, so that we may replace (3) by the following equivalent set of conditions:

$$\int_a^b p(x)x^kQ_n(x)dx=0; \quad k=0,1,\cdots,n-1; \quad n=1,2,\cdots. \tag{4}$$

We readily prove now that (4) is satisfied by the polynomial

$$Q_n(x)=\begin{vmatrix} \alpha_0 & \alpha_1\cdots & \alpha_n \\ \alpha_1 & \alpha_2\cdots & \alpha_{n+1} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \alpha_{n-1} & \alpha_n\cdots & \alpha_{2n-1} \\ 1 & x\cdots & x^n \end{vmatrix}; \quad n=1,2,\cdots; \quad Q_0=1. \tag{5}$$

In fact, multiply the last row of (5) by x^k, substitute in (4) and integrate by the familiar rule (i. e. the last row). We get, by the definition of α_i, the determinant

$$\begin{vmatrix} \alpha_0 & \alpha_1\cdots & \alpha_n \\ \alpha_1 & \alpha_2\cdots & \alpha_{n+1} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \alpha_{n-1} & \alpha_n\cdots & \alpha_{2n-1} \\ \alpha_k & \alpha_{k+1}\cdots & \alpha_{n+k} \end{vmatrix}$$

and this obviously vanishes, if $k=0,1,\cdots,n-1$, for then two rows become identical. Furthermore, $Q_n(x)$ is precisely of degree n, for the coefficient of x^n in (5) is the determinant

$$\begin{vmatrix} \alpha_0 & \alpha_1\cdots & \alpha_{n-1} \\ \alpha_1 & \alpha_2\cdots & \alpha_n \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \alpha_{n-1} & \alpha_n\cdots & \alpha_{2n-2} \end{vmatrix},$$

which is always *positive*, due to the non-negativeness of $p(x)$ in (a,b). Evidently the set $\{C_nQ_n(x)\}$, with arbitrary constant factors C_n, is also a set of *OP*. Disregarding such constant factors, the *set* $\{Q_n(x)\}$ *is*

unique. We prove this as follows. Let $G_s(x) = g_0 + g_1 x + \cdots + g_s x^s$ denote an *arbitrary* polynomial, of degree $\leqq s$. Replace (4) by the following equivalent set of orthogonality conditions, very useful in applications:

$$\int_a^b p(x)Q_n(x)G_{n-1}(x)dx = 0; \quad n = 1,2,\cdots. \tag{6}$$

Assume now the existence of a second *OP*, say, $R_n(x)$, of the same degree n, distinct from $Q_n(x)$. If $Q_n(x) = q_n x^n + \cdots$, $R_n(x) = z_n x^n + \cdots$, then

$$S(x) = z_n Q_n(x) - q_n R_n(x) \qquad (z_n q_n \neq 0)$$

is a polynomial *of degree* $< n$. Now, by hypothesis,

$$I_1 = \int_a^b p(x)Q_n G_{n-1}(x)dx = 0, \quad I_2 = \int_a^b p(x)R_n(x)G_{n-1}(x)dx = 0,$$

whence

$$z_n I_1 - q_n I_2 = \int_a^b p(x)S(x)G_{n-1}(x)dx = 0$$

for an *arbitrary* polynomial $G_{n-1}(x)$, of degree $\leqq n-1$. Take here $G_{n-1}(x) \equiv S(x)$, and we get

$$\int_a^b p(x)S^2(x)dx = 0,$$

which is impossible, the integrand being non-negative (unless $S(x) \equiv 0$).

Remark. Our set of *OP* becomes completely determined if we specify it in a certain manner. The customary specification is one of the following:

a) Make the highest coefficient in each *OP* equal unity. We get from (5):

$$\Phi_n(x) = \begin{vmatrix} \alpha_0 & \alpha_1 & \cdots \alpha_n \\ \alpha_1 & \alpha_2 & \cdots \alpha_{n+1} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \alpha_{n-1} & \alpha_n & \cdots \alpha_{2n-1} \\ 1 & x & \cdots x^n \end{vmatrix} : \begin{vmatrix} \alpha_0 & \alpha_1 & \cdots \alpha_{n-1} \\ \alpha_1 & \alpha_2 & \cdots \alpha_n \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \alpha_{n-1} & \alpha_n & \cdots \alpha_{2n-2} \end{vmatrix} \tag{7$_a$}$$

$$= x^n + \cdots. \quad (n \geqq 1; \quad \Phi_0 = 1).$$

b) "Normalize" the set $\{\Phi_n(x)\}$, taking

$$(7_b) \qquad \varphi_n(x)=\Phi_n(x) \Big/ \sqrt{\int_a^b p(x)\Phi_n^2(x)dx}\ ; \qquad n=0,1,2,\cdots,$$

so that

$$(8) \qquad \begin{cases} \displaystyle\int_a^b p(x)\varphi_m(x)\varphi_n(x)dx=0; \quad m,n=0,1,2,\cdots; \quad m\neq n \\ \displaystyle\int_a^b p(x)\varphi_n^2(x)dx=1; \quad n=0,1,2,\cdots. \end{cases}$$

The set $\{\varphi_n(x)\}$ constitutes an "orthonormal" set of polynomials corresponding to the given interval (a,b) and to the "weight-function" $p(x)$. If a function $f(x)$ is given, we have now the formal expansion (making use of(8))

$$(*) \qquad f(x)\sim\sum_{n=0}^{\infty} f_n\varphi_n(x), \qquad \int_a^b p(x)\varphi_n(x)f(x)dx,$$

and we recognize at once the analogy of this expansion to the Fourier (trigonometric) series.

The orthogonality property (6) or either of its equivalents (3), (4), completely characterizes the *OP* set $\{\Phi_n(x)\}$. An all-powerful mind could have said: If you give me the weight-function $p(x)$ and the orthogonality property (6), then I know everything about the corresponding *OP* set. In some respects we can go further, even without knowing the explicit expression of $p(x)$ (or its moments). In fact, there are many features common to *all* sets of *OP*, features arising directly from the orthogonality property. We proceed to prove the following theorem of fundamental importance in the theory of *OP* and its applications. It deals with the "zeros" of $\Phi_n(x)$, i. e. with the roots of the equation $\Phi(x)=0$.

Theorem on the zeros of OP. The roots of the equation $\Phi_n(x)=0(n>I)$ are real and distinct; they lie inside (a,b).

$\Phi_n(x)$ being a polynomial of degree n, it suffices to prove that it changes sign inside (a,b) precisely n times. Assume the contrary: $\Phi_n(x)$ changes sign at m interior points $c_1,c_2,\cdots\cdots,c_m$, where $O\leqq m<n$. Then $\pm\Phi_n(x)(x-c_1)(x-c_2)\cdots(x-c_m)$, with a proper choice of $\pm$, is

never negative in (a,b) (we replace the product $(x-c_1)\cdots(x-c_m)$ by unity, if $m=0$), so that by

$$\int_a^b p(x)\Phi_n(x)[\pm(x-c_1)\cdots(x-c_m)]dx>0,$$

which, if $m\leqq n-1$, contradicts (6), where we take

$$G_{n-1}(x)=\pm(x-c_1)\cdots(x-c_m).$$

Geometrically it means that the parabola $y=\Phi_n(x)$ crosses the x-axis precisely n times between a and b (n "nodes"). It thus resembles the curve $y=\cos nx$ between 0 and π. In many cases this resemblance is not a superficial, but a far-reaching property of *OP*. We shall denote the zeros of $\Phi_n(x)$ as follows:

$$x_{1,n}<x_{2,n}<\cdots<x_{n,n}, \tag{9}$$

writing simply x_i in place of $x_{i,n}$ if there is no danger of confusion. We shall also write

$$\varphi_n(x)=a_n\Phi_n(x)=a_nx^n+a_{n,n-1}x^{n-1}+\cdots,a_n>0\text{—} \tag{10}$$

"normalizing coefficient".

We may add that the zeros of $\Phi_n(x)$ and $\Phi_{n+1}(x)$ $(n\geqq 1)$ interlace so that the zeros of $\Phi_n(x)$ spread out, as n increases, i. e.

$$x_{1,n+1}<x_{1,n},\quad x_{n,n}<x_{n+1,n+1},$$

which leads to the conclusion—generally correct—that the extreme zeros $x_{1,n}$, $x_{n,n}$ tend respectively to the end-points a,b, as $n\to\infty$.

2.2 *Illustrations.* (Cf. Jackson, l. c.).

(i) (a,b) finite, say, $(-1,1)$, $p(x)=1$. This yields *Legendre Polynomials*—historically the oldest and the most important set of *OP* (Theory of Potential, Spherical Harmonics). Here, in place of the general expression (5), we can use the more convenient one

$$\begin{cases} P_n(x)=\dfrac{1}{2^n\cdot n!}\,\dfrac{d^n[(x^2-1)^n]}{dx^n}{}^{*} \\[2ex] \qquad =\dfrac{1\cdot 3\cdots(2n-1)}{n!}\left[x^n-\dfrac{n(n-1)}{2\cdot(2n-1)}x^{n-2}\pm\cdots\right] \\[2ex] P_0=1,\quad P_1=x,\quad P_2=3/2(x^2-1/3),\cdots. \end{cases} \tag{11}$$

$$(12)\qquad \int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}\,,$$

$$a_n = \frac{1\cdot 3\cdots(2n-1)}{n!}\sqrt{\frac{2n+1}{2}}\quad (n=0;\ 0!=1),$$

so that the orthonormal set of Legendre Polynomials is

$$(13)\quad \varphi_n(x) = \sqrt{\frac{2n+1}{2}}\,P_n(x)$$

$$= \frac{1\cdot 3\cdots(2n-1)}{n!}\sqrt{\frac{2n+1}{2}}\left[x^n - \frac{n(n-1)}{2\cdot(2n-1)}x^{n-2} \pm \cdots\right],\quad n \geqq 0.$$

(ii) $(a,b)=(-1,\ 1)$, $p(x)=(1-x^2)^{-\frac{1}{2}}$. This leads to *Trigonometric* (often called "Tchebycheff") orthonormal *Polynomials*:

$$(14)\quad \varphi_0(x) = \sqrt{\frac{1}{\pi}}\ ;\quad \varphi_n(x) = \sqrt{\frac{2}{\pi}}\ \cos(n \text{ arc} \cos x)$$

$$= 2^{n-1}\sqrt{\frac{2}{\pi}}\left(x^n - \frac{n}{1\cdot 4}\ x^{n-2} \pm \cdots\right).$$

$$a_n = 2^{n-1}\sqrt{\frac{2}{\pi}}\ ,\quad n \geqq 1.$$

The orthogonality property of the polynomials (14) is verified at once by letting in (3) $x=\cos\varphi$.

Both Legendre and trigonometric polynomials are special cases of the so-called *Jacobi Polynomials* corresponding to

$$(a,b)=(-1,1);\quad p(x)=(1+x)^{\alpha-1}(1-x)^{\beta-1},\quad \alpha,\beta>0.$$

*This expression naturally arises in the Potential Theory from the expansion

$$(1-2tx+t^2)^{\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

The orthogonality property

$$(3)\qquad \int_{-1}^{1} P_m(x)P_n(x)dx = 0 \qquad (m \neq n)$$

can be easily verified by applying a repeated integration by parts to

$$\int_{-1}^{1} \frac{d^m[(x^2-1)^m]}{dx^m}\cdot\frac{d^n[(x^2-1)^n]}{dx^n}\,dx$$

The same procedure yields, for $m=n$, formula (12).

(iii) $(a,b)=(0,\infty)$, $p(x)=e^{-x}$—*Laguerre Polynomials*

$$(15)\quad \begin{cases} \varphi_n(x)=\dfrac{1}{n!}\,e^{x}\dfrac{d^n(e^{-x}x^n)}{dx^n} \end{cases}$$

$$=\frac{1}{n!}\left\{x^n-\frac{n^2}{1}x^{n-1}+\frac{n^2(n-1)^2}{1\cdot 2}\,x^{n-2}\pm\cdots\right\},$$

$$a_n=\frac{1}{n!},\quad n\geqq 0.$$

(iv) $(a,b)=(-\infty,\infty)$, $p(x)=e^{-x^2}$—*Hermite Polynomials.*

$$(16)\quad \begin{cases} \varphi_n(x)=\dfrac{1}{\sqrt[4]{\pi}}\sqrt{\dfrac{2^n}{n!}}\cdot\dfrac{1}{(-2)^n}\,e^{x^2}\,\dfrac{d^n(e^{-x^2})}{dx^n} \\[2ex] \quad =\sqrt{\dfrac{2^n}{\sqrt{\pi}\cdot n!}}\left\{x^n-\dfrac{1}{4}\dfrac{n(n-1)}{1}\,x^{n-2}\pm\cdots\right\}, \\[2ex] a_n=\sqrt{\dfrac{2^n}{\sqrt{\pi}n!}},\quad n\geqq 0. \end{cases}$$

(concerning orthogonality of (15, 16) see preceding footnote). Laguerre and Hermite Polynomials play an important role in Theoretical Physics and in Statisitics. Regarding Trigonometric Polynomials one recognizes their relationship to Fourier Series.

2.3. *Mechanical quadratures.* The discussion of the expansion (*) lies outside the scope of the present paper. We turn to another very important application of *OP*. It deals with the approximate evaluation of definite integrals, what is known as "mechanical quadratures".

Any polynomial of degree $n-1$

$$G_{n-1}(x)=g_0+g_1x+\cdots+g_{n-1}x^{n-1}$$

is completely determined if its values are given at n arbitrarily preassigned distinct points $c_1,c_2,\cdots,c_n$:

$$(17)\qquad G_{n-1}(c_i)=y_i,\quad i=1,2,\cdots,n,$$

for the system (17), where the c_i and the y_i are known, determines completely the coefficients g_i. This further yields a unique value for the integral

$$\int_a^b p(x)G_{n-1}(x)dx,$$

where $p(x)$ is a given function. We thus may expect the value of this integral to be expressible in terms of the quantities $G_{n-1}(c_i)$. The simplest expression of this kind is a *linear* one:

$$\int_a^b p(x)G_{n-1}(x)dx = \sum_{i=1}^{n} l_i G_{n-1}(c_i), \tag{18}$$

where *the coefficients* l_i *are independent of* $G_{n-1}(x)$, for the c_i may be preassigned in advance, once for all $G_{n-1}(x)$. What is the necessary expression for l_i? To answer this, apply (18) to the special polynomial

$$L_i(x) = \frac{(x-c_1)(x-c_2)\cdots(x-c_{i-1})(x-c_{i+1})\cdots(x-c_n)}{(c_i-c_1)(c_i-c_2)\cdots(c_i-c_{i-1})(c_i-c_{i+1})\cdots(c_i-c_n)} \tag{19}$$

which can be written as

$$L_i(x) = \frac{\varphi(x)}{(x-c_i)\varphi'(c_i)}, \quad \varphi(x) = (x-c_1)(x-c_2)\cdots(x-c_n). \tag{20}$$

We see at once that

$$L_i(c_i) = 1; \quad L_i(c_j) = 0, \quad j \neq i,$$

so that the right-hand side of (18) reduces to a single term, namely,

$$\int_a^b p(x)L_i(x)dx = l_i.$$

This necessary expression for l_i, $i = 1,2,\cdots,n$, is also sufficient as is proved on the basis of the Lagrange Interpolation Formula. We now have for any polynomial of degree $n-1$

$$\int_a^b p(x)G_{n-1}(x)dx \tag{21}$$

$$= \sum_{i=1}^{n} l_i G_{n-1}(c_i), \qquad l_i = \int_a^b \frac{p(x)\varphi(x)}{(x-c_i)\varphi'(c_i)} dx.$$

In applications we apply (21) to a continuous function $f(x)$ defined on (a,b). Here we naturally restrict the points c_i to belong to (a,b), and in place of the exact formula (21) we obtain an approximate formula

$$(22)\qquad \int_a^b p(x)f(x)dx \sim \sum_{i=1}^{n} l_i f(c_i), \qquad l_i = \int_a^b \frac{p(x)\varphi(x)}{(x-c_i)\varphi'(c_i)}\,dx.$$

The terminology "mechanical quadratures" may be explained by the fact that the quantities $f(c_i)$ in (22) can be bound by mechanical means if $f(x)$ is given graphically.

Assume now the points c_i are chosen from an infinite triangular array

$$\begin{array}{lll} c_{1,1} & & \\ c_{1,2}, & c_{2,2} & \\ c_{1,3}, & c_{2,3}, & c_{3,3} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots . \end{array}$$

Recalling Weierstrass' Approximation Theorem, we are led to expect that in (22), where (a,b) is finite, the right-hand member tends to its left-hand member as a limit, for $n\to\infty$, provided, *the above array of points is properly chosen.* Then we say: *the mechanical quadratures formula* (22) converges.

The great Gauss, "king of mathematicians" (1777-1855), more than a hundred years ago (1816) conceived the ingenious idea of *choosing the n points c_i and the n coefficients l_i in* (22) *so that the formula shall be exact for any polynomial of degree* $\leqq 2n-1$. (We cannot go higher, for the $2n$ unknown quatities $c_1, c_2, \cdots, c_n$ and $l_1, l_2, \cdots l_n$ must now satisfy $2n$ equations, namely:

$$\int_a^b p(x)x^k dx = \sum_{i=1}^{n} l_i c_i^k; \quad k=0,1,2,\cdots, 2n-1).$$

Gauss considers the special case—(a,b) finite, $p(x)\equiv 1$; we shall consider the most general case, where (a,b) is any given interval, finite or infinite, and $p(x)$ is defined on (a,b) (naturally all integrals

$$\int_a^b p(x)x^n dx, \quad n=0,1,2,\cdots, \text{ exist}).$$

The following remarkable fact is revealed in the next

Theorem. *If $p(x)$ is a weight-function defined on (a,b) then with $c_i = x_i$, $(i=1,2,\cdots,n)$—zeros of the corresponding OP, $\varphi_n(x)$, Gauss' requirements are satisfied, that is*

$$(23)\qquad \int_a^b p(x)G_{2n-1}(x)dx=\sum_{i=1}^n H_iG_{2n-1}(x_i)^*$$

$$(24)\qquad H_i=\int_a^b \frac{p(x)\varphi_n(x)}{(x-x_i)\varphi_n'(x_i)}\,dx.$$

This choice of the c_i is the only possible one, for a given $p(x)$. Moreover, the coefficients H are positive and bounded for all n; we have

$$(25)\qquad \int_a^b p(x)dx=H_1+H_2+\cdots+H_n;\ =H_i<\int_a^b p(x)dx.\quad i=1,2,\cdots,n.$$

Proof. Divide $G_{2n-1}(x)$ by $\varphi_n(x)$, we get:

$$G_{2n-1}(x)=\varphi_n(x)G_{n-1}(x)+G_{n-1}(x),$$

$$G_{2^n-1}(x_i)=\varphi_i(x_i)G_{n-1}(x_i)+G_{n-1}(x_i)=G_{n-1}(x_i),\quad i=1,2,\cdots,n.$$

Hence, making use of (6) and (21), where we replace c_i by x_i and l_i by H_i,

$$\int_a^b p(x)G_{2n-1}(x)dx=\int_a^b p(x)\varphi_n(x)G_{n-1}(x)dx$$

$$+\int_a^b p(x)G_{n-1}(x)dx=\int_a^b p(x)G_{n-1}(x)dx$$

$$=\sum_{i=1}^n H_iG_{n-1}(x_i)=\sum_{i=1}^n H_iG_{2n-1}(x_i).$$

Thus, the choice $c_i=x_i$ $(i=1,2,\cdots,n)$ is sufficient for our purpose; its necessity is proved in the same manner. We can apply now (23) to the polynomial

$$\left[\frac{\varphi_n(x)}{(x-x_i)\varphi_n'(x_i)}\right]^2,$$

of degree $2n-2$. Note (see the foregoing discussion of $L_i(x)$) that

$$\left[\frac{\varphi_n(x)}{(x-x_i)\varphi_n'(x_i)}\right]^2=1 \text{ for } x=x_i,\quad =0 \text{ for } x=x_j,\ j\neq i.$$

*In the special case considered by Gauss, formula (23) employs Legendre Polynomials.

This gives: $$H_i = \int_a^b p(x) \left[\frac{\varphi_n(x)}{(x-x_i)\varphi_n'(x_i)} \right]^2 dx > 0; \quad i = 1,2,\cdots, n.$$

Finally, $$\int_a^b p(x)\cdot 1 \cdot dx = H_1 + H_2 + \cdots + H_n.$$

The simplest case of (23) is that of Trigonometric Polynomials. Here all H_i are equal, so that, by (25),

$$H_1 = H_2 = \cdots = H_n = \frac{1}{n}\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{n}.$$

The foregoing properties of the coefficients H_i in (23) are of fundamental importance. They yield at once, on the basis of Weierstrass' Theorem, the *convergence as $n \to \infty$, of the mechanical quadratures formula,*

$$(^{**}) \qquad \int_a^b p(x) f(x) dx \sim \sum_{i=1}^{n} H_i f(x_i)$$

for any finite interval (a,b) and for any weight-function $p(x)$, if $f(x)$ is integrable, and a fortiori, continuous, in (a,b).

This remarkable theorem was proved in 1884 by the French-Dutch mathematician Stieltjes (1856-1894) in a classical paper on mechanical quadratures. It makes formula (**) especially important in applications.

Viewed geometrically, formula (**) is striking. It tells us that *all parabolas $y = g_0 + g_1 x + \cdots + g_{2n-1}x^{2n-1}$ passing through the same n fixed points $\{x_i, y_i\}$, $i = 1,2,\cdots, n$, where the x_i are the zeros of $\varphi_n(x)$ and the y_i are preassigned arbitrarily, yield the same value for the integral*

$$\int_a^b p(x) y dx,$$

however varied may be the coefficients $g_0, g_1, \cdots, g_{2n-1}$. The restriction $p(x) \geqq 0$ in (a,b) *is essential*; dropping it, we cannot in general satisfy Gauss' requirements.

3. *Application of OP to polynomials in general.*

3.1. *Estimate of a linear combination of the coefficients of a polynomial.*

Problem. *Given a function $q(x)$ defined over a certain interval (a,b), finite or infinite, and a set of real constants $M>0, \alpha_0, \alpha_1, \cdots, \alpha_n$.* Find an *upper bound for the absolute value of the linear combination*

$$\omega(G_n)=\alpha_0 g_0+\alpha_1 g_1+\cdots+\alpha_n g_n$$

of the coefficients, for all polynomials $G_n(x)=g_0+g_1x+\cdots+g_nx^n$ *such that*

$$|q(x)G_n(x)|\leqq M, \quad a\leqq x\leqq b. \tag{26}$$

Solution. Assume we can find a non-negative function $r(x)$ such that

$$\int_a^b z(x)dx \qquad \text{exists, and}$$

$$p(x)=q^2(x)z(x) \tag{27}$$

is a weight-function in (a,b). Introduce the set $\{\varphi_n(x)\}$ of orthonormal polynomials corresponding to (a,b) and $p(x)$. Then

$$|\omega(G_n)|\leqq\sqrt{\int_a^b p(x)G_n^2(x)dx\cdot\sum_{i=0}^{n}\omega^2(\varphi_i)}. \tag{28}$$

We omit the proof)*. Formula (28) is *fundamental* and solves our problem, namely,

$$|\omega(G_n)|\leqq M\sqrt{\int_a^b z(x)dx\cdot\sum_{i=0}^{n}\omega^2(\varphi_i)} \quad \text{(under condition (26))}. \tag{29}$$

*Cf. J. Shohat: On a General Formula...., Transactions Amer. Math. Soc., v. 29 (1927), pp. 569-583.

Note the generality of our problem and of its solution. By specifying the α_i, $q(x)$, $z(x)$, we obtain a great variety of estimates for polynomials.

Illustration. Consider the class of polynomials

$$G_n(x)=g_0+g_1x+\cdots+g_{n-1}x^{n-1}+g_nx^n,$$

n given, for which one of the following inequalities holds:

$$\left\{\begin{array}{lll} a) & |G_n(x)|\leqq M, & -1\leqq x\leqq 1. \\ b) & e^{-x/4}|G_n(x)|\leqq M, & 0\leqq x<\infty \\ c) & e^{-x^2/4}|G_n(x)\leqq M, & -\infty<x<\infty. \end{array}\right. \tag{30}$$

Find estimates for the coefficients g_n, g_{n-1}.

These estimates are obtained at once from our general formula (29), where we take $\alpha_n = 1$, $\alpha_{n-1} = \alpha_{n-2} = \cdots = \alpha_0 = 0$, when dealing with g_n, and $\alpha_{n-1} = 1, \alpha_n = \alpha_{n-2} = \cdots = \alpha_0 = 0$, when dealing with g_{n-1}. (29) becomes correspondingly. (See form. (10)):

$$(31)\quad \begin{cases} |g_n| \leqq a_n \sqrt{\displaystyle\int_a^b p(x) G_n{}^2(x)dx} \\[2ex] |g_{n-1}| \leqq \sqrt{a^2_{n-1} a^2_{n,n-1}} \cdot \sqrt{\displaystyle\int_a^b p(x) G_n{}^2(x)dx,} \end{cases} \qquad (p(x) = q^2(x) z(x))$$

whence, *under condition* (26),

$$(32)\quad \begin{cases} |g_n| \leqq a_n M \sqrt{\displaystyle\int_a^b z(x)dx} \\[2ex] |g_{n-1}| \leqq \sqrt{(a_n{}^2{}_{-1} + a^2_{n,n-1}) \cdot \displaystyle\int_a^b z(x)dx \cdot M.} \end{cases}$$

Thus, for (30a) we take in (32) $q(x) = 1$, $z(x) = (1-x^2)^{-\frac{1}{2}}$, and make use of the Trigonometric Polynomials (14). This gives:

(33a) $|G_n(x)| \leqq M_{in}(-1,1)$ implies:

$$|g_n| \leqq 2^{n-1} \sqrt{\frac{2}{\pi} \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}} = 2^{n-\frac{1}{2}}; \quad |g_{n-1}| \leqq 2^{n-5/2}.$$

For (30 b,c) take in (32) correspondingly

$$z(x) = e^{-x/2}, \quad p(x) = e^{-x/2} e^{-x/2} = e^{-x}; \quad z(x) = e^{-x^2/2}, \quad p(x) = e^{-x2}$$

and use the polynomials of Laguerre and Hermite (form. (15,16)). We get:

(33b) $e^{-x/4} |G_n(x)| \leqq M_{in}(0, \infty)$ implies:

$$|g_n| \leqq \frac{1}{n!} M \sqrt{\int_0^\infty e^{-x/2} dx} = \frac{M\sqrt{2}}{n!} \quad ; \quad |g_{n-1}| \leqq \frac{M\sqrt{2}}{(n-1)!} \sqrt{n^2+1.}$$

(33c) $e^{-x^2/2} |G_n(x)| \leqq M_{in}(-\infty, \infty)$ implies:

$$|g_n| \leqq M \sqrt{\frac{2^n}{\sqrt{\pi} \cdot n!} \cdot \int_{-\infty}^{\infty} e^{-x^2/2} dx}$$

$$=\sqrt{\frac{2^{n+1/2}}{n!}}M; \quad |g_{n-1}|\leqq\sqrt{\frac{2^{n-1/2}}{(n-1)!}}M$$

(recalling that $$\int_{-\infty}^{\infty}e^{-y^2}dy=\sqrt{\pi}).$$

Similarly, taking $\omega(G_n)\equiv G_n'(z)$, z denoting a certain fixed point, we derive from (29) the following result:

$$(34)\quad |G_n'(z)|\leqq M\sqrt{\int_a^b z(x)dx\cdot\sum_{i=1}^{n}[\varphi_i'(z)]^2}\quad \text{(under condition (26))},$$

and this furnishes an estimate for the derivative $G_n'(z)$ if we know—and we do know in many cases—an estimate for

$$\sum_{i=1}^{n}[\varphi_i'(z)]^2.$$

Remarks. (i) Using special methods, adapted ad hoc to each particular case (mainly for a finite interval), better estimates have been obtained for $|g_n|, |g_{n-1}|$, under condition (26). Our aim was to show that all above estimates stem from a *single source*—formula (29).

(ii) We note that our estimate for $|\omega(G_n)|$ depends upon the value of

$$\int_a^b p(x)G_n^2(x)dx,$$

and this enhances its applicability, for the latter integral occurs frequently and may be known from the nature of the particular problem under consideration, even if the inequalities in (30) are not available. Moreover, using (28) in place of (29), we can improve our estimates as follows. Apply to

$$\int_a^b p(x)G_n^2(x)dx$$

the mechanical quadratures formula (23), where n is replaced by $n+1$, and write $x_{i,n+1}$ ($i=1, 2,\cdots,n+1$) for the zeros of $\varphi_{n+1}(x)$ and $H_{i,n+1}$ in place of H_i. We get:

$$(35)\qquad |\omega(G_n)| \leqq \sqrt{\sum_{i=1}^{n+1} H_{i,n+1} G_n^2(x_{i,n+1}) \cdot \sum_{i=0}^{n} \omega^2(\varphi_i)}.$$

Now, let M' denote the largest of the quantities $G_n^2(x_{1,n+1}), G_n^2(x_{2,n+1}), \cdots, G_n^2(x_{n+1\,n+1})$. Formula (35, by virtue of (25), gives:

$$(36)\qquad |\omega(G_n)| \leqq M' \sqrt{\int_a^b p(x)dx \sum_{i=0}^{n} \omega^2(\varphi_i)},$$

and this again yields a great variety of estimates for polynomials. These estimates are better than those obtained above for in general $M' < M$.

To illustrate, take $(a,b) = (-1,1)$, $p(x) = (1-x^2)^{-1/2}$. Here $\varphi_{n+1}(x)$ is the trigonometric polynomial (14) $\sqrt{2/\pi}\cos[(n+1)\text{arc}\cos x]$, whose zeros are

$$\cos\frac{\pi}{2(n+1)},\ \cos\frac{3\pi}{2(n+1)},\cdots,\ \frac{\cos(2n+1)\pi}{2(n+1)},$$

and we get the following result—an improvement over (33a): if the values taken by a polynomial $g_0 + g_1x + \cdots + g_nx_n$, of degree n, at the points

$$\cos\frac{\pi}{2n+2},\quad \cos\frac{3\pi}{2n+2},\quad \cos\frac{(2n+1)\pi}{2n+2};$$

do not exceed numerically a given constant M', then $|g_n| < 2^{n-1/2}M'$.

3.2. *A more precise form, for polynomials, of the Mean-Value Theorem and of Rolle's Theorem,* We are familiar with the following fundamental statements for continuous functions:

(i) $f(b) - f(a) = (b-a)f'(\alpha)$, α in (a,b) finite.

(ii) $f(a) = f(b)$ implies $f'(\alpha) = 0$, α in (a,b).

(iii) If $p(x) > 0$ in (a,b), then

$$\int_a^b p(x)f(x)dx = \mu \int_a^b p(x)dx,$$

where μ is a certain quantity lying between the smallest and largest value of $f(x)$ in (a,b).

(iv) $$\int_a^b p(x)f(x)dx = 0,$$

where $p(x)$ is non-negative, implies $f(x)$ vanishes at a point α in (a,b). (Of course, α is different in different formulae).

One may expect a more precise characterization of α and μ in the foregoing statements if one narrows the class of functions $f(x)$ to polynomials. Such is actually the case. Again one shows this very simply, on the basis of the mechanical quadratures formula (23).

We start with (i), where we assume, for the sake of simplicity $(a,b)=(-1,1)$. Let $f(x)$ be a polynomial, of degree $2n$ or $2n-1$. Writ,

$$f(1)-f(-1)=\int_{1}^{1} f'(x)dx.$$

On the right side we may apply the mechanical quadratures formula (23) with $p(x)=1$. This gives

$$f(1)-f(-1)=H_1f'(x_1)+H_2f'(x_2)+\cdots+H_nf'(x_n), \tag{37}$$

$x_i(i=1,2,\cdots,n)$ denoting the zeros of the Legendre polynomials $P_n(x)$ (form. (11)). Denote by m,M respectively the smallest and largest of the quantities $f(x_1)f(x_2),\cdots,f(x_n)$. It follows, all H_i being positive

$$m\sum_{i=1}^{n} H_i \leqq \sum_{i=1}^{n} H_if(x_i) \leqq M\sum_{i=1}^{n} H_i,$$

whence, by (25),

$$f(1)-f(-1)=\mu\int_{-1}^{1} dx=2\mu, \quad m\leqq\mu\leqq M.$$

We thus get for polynomials an improved version of the Mean-Value Theorem of Differential Calculus, namely: *if $f(x)$ is a polynomial, of degree $2n$ or $2n-1$, then $f(1)-f(-1)=2\mu$, where μ lies between the smallest and largest value taken by $f(x)$ at the n zeros of the Legendre polynomiat $P_n(x)$.*

(ii) Can be treated in the same manner, by means of (37). We get the following improved version of Rolle's Theorem for polynomials. *If a polynomial $f(x)$, of degree $2n$ or $2n-1$, takes the same values at $x=-1$ and $x=1$, then its derivative vanishes in the interval formed by the extreme zeros of the Legendre polynomial $P_n(x)$.*

Turning to (iii), where $f(x)$ is a polynomial of degree $2n-1$ or $2n-2$, write, again using (23),

$$\int_a^b p(x)f(x)dx=\sum_{i=1}^{n} H_if(x_i),$$

and reasoning as above in (i), we get

$$m\sum_{i=1}^{n} H_i \leqq \int_a^b p(x)f(x)dx \leqq M\sum_{i=1}^{n} H_i,$$

m,M denoting respectively the smallest and largest of the quantities $f(x_1), f(x_2), \cdots, f(x_n)$, the x_i being the zeros of $\varphi_n(x)$. This leads, through (25), to the following improved version for polynomials of the Mean-Value Theorem of Integral Calculus.

If $f(x)$ is a polynomial of degree $2n-1$ or $2n-2$, then

$$\int_a^b p(x)f(x)dx = \mu \int_a^b p(x)dx \quad (p(x) \geqq 0)$$

*where μ lies between the smallest and largest value taken by $f(x)$ at the n zeros of $\varphi_n(x)$—OP corresponding to (a,b) and $p(x)$.**

In particular

$$\int_1^1 f(x)dx = \mu \int_1^1 dx = 2\mu,$$

μ being intermediate between the values taken by $f(x)$, polynom of degree $2n-1$ or $2n-2$, at the n zeros of $P_n(x)$.

To illustrate: $$\int_1^1 (ax^3+bx^2+cx+d)dx = 2\mu,$$

μ being intermediate between the values taken by the integrand at the points $x = -\sqrt{3}/3,\ 0,\ \sqrt{3}/3$— the zeros of $P_2(x) = 3/2(x^2-1/3)$. The above discussion of (iii) has prepared us for the treatment of (iv), and we state its improved version for polynomials as follows.

If $f(x)$ is a polynomial of degree $2n-1$ or $2n-2$, and

$$\int_a^b p(x)f(x)dx = 0, \quad p(x) \geqq 0,$$

then $f(x)$ vanishes in the interval formed by the extreme zeros of $\varphi_n(x)$.†
For example,

*If the degree of $f(x)$ is less than $2n-2$, then we use $\varphi_m(x)$, with a properly chosen $m < n$.

†A more comprehensive article by J. R. Kline, summarizing the work of Professor Shohat, was published in *Science* shortly after his death.

$$\int_{-1}^{1}(ax^3+bx^2+cx+d)dx=0$$

implies the integrand vanishes at least once in the interval

$$\left(-\frac{\sqrt{3}}{3},\ \frac{\sqrt{3}}{3}\right),$$

and this is better than the general statement that it vanishes somewhere in $(-1,1)$.

The author ventures to express the hope that the foregoing discussion, of necessity restricted and sketchy, throws some light on the important role played by orthogonal polynomials in the general theory of polynomials.

On the Structure of Certain Tensors

by H. V. CRAIG

1. *Introduction.* The primary object of this paper is to express the higher order intrinsic derivatives of absolute vectors as contractions of extensors.* We take the view that the structure of a tensor is known if it is recognized that the tensor is built up from other tensors and extensors (the basic extensors) by the elementary processes addition, multiplication, and contraction. Thus what we have in mind might very well be called algebraic structure.

The decomposition of a tensor into structural elements is not necessarily unique and as we shall see may be accomplished in a trivial and ad hoc fashion. A decomposition is interesting and of aesthetic value if the basic constituents possess invariance of functional form. A decomposition is of particular value, from the standpoint of the theory of invariance, if it discloses the hitherto unknown fact that a certain collection of quantities which are already known to be of importance in some other respect constitute the components of an extensor. Such a

*The corresponding problem for weighted vectors has been treated by J. M. Hurt in a Master's thesis.

discovery opens the way for the systematic construction of other extensors, tensors, and invariants which are invariant in functional form.

As an illustration of a trivial and unexciting decomposition, let us consider the following solution for the problem of expressing a given invariant I as a contraction of tensors. First we select a coordinate system x and define a contravariant vector V by assigning V^a the value $\delta_1{}^a I$ in system x. The components of V in the other coordinate systems are to be determined of course by the tensor transformation law. Similarly, let a covariant vector U be determined by the equation $U_a = \delta_a{}^1$. Obviously, $V^a U_a = I$ and this equation is valid in all coordinate systems. The defect is, of course, that the vectors U and V lack invariance of functional form.

As a contrasting case, let us assume that we have followed the classical procedure and proved that the intrinsic derivative $\delta V^a/\delta t$ of a contravariant vector V^a is again a contravariant vector. That is, we assume that we have discovered the Christoffel symbols, established their transformation law, differentiated the transformation equation for V^a, eliminated the unwelcome second derivatives by means of the affine connection, etc. That is, supposedly we have proved by direct calculation that the quantities $V'^a + V^b\left\{{a \atop b}\right\}$, $\left\{{a \atop b}\right\} = \left\{{a \atop bc}\right\} x'^c$ are the components of a tensor (the primes indicate differentiation). We are now in possession of the corner stone for some celebrated mathematics but from our point of view the structure of the intrinsic derivative is still unknown. The problem of establishing the structure is however exceedingly simple. The quantities V^a, V'^a are the components of an extensor for arbitrary V^a, and $\delta V^a/\delta t$ may be written in the form $V^{\beta b} g^a{}_{\beta b}$ with β summed from zero to one. Here V^{0b} and V^{1b} denote V^b and V'^b, while $g^a{}_{1b} = \delta^a{}_b$ and $g^a_{0b} = \left\{{a \atop b}\right\}$. From the arbitrariness of V^b and the tensor character of $\delta V^a/\delta t$, we may conclude that the $g^a{}_{\beta b}$ are the components of an extensor.* Our new extensor $g^a{}_{\beta b}$ or $\delta^a{}_b$, $\left\{{a \atop b}\right\}$ does possess invariance of functional form in terms of the Kronecker deltas and the two index Christoffel symbols. Also, the latter are linear forms in the x''s in all coordinate systems. Thus having discovered a new extensor, we can construct other extensors, tensors, and invariants at once.

Perhaps we should remark in passing, that in the case of a Riemann space embedded in a Euclidean space, it is quite obvious from one point of view that the quantities $\delta^a{}_b$, $\left\{{a \atop b}\right\}$ are the components of an extensor. Once this has been noted the tensor character of the intrinsic derivative is self evident and the affine connection as an *isolated* entity loses its

*We shall give presently a resume of certain aspects of the theory of extensors. For a more detailed account see numbers 1, 2, 3, of the appended bibliography.

importance. In the case of the geometry of paths, one can prove readily that if $L^a{}_b$ is a linear form in the x''s and the quantities $\delta^a{}_b$, $L^a{}_b$ are adopted as the components of an extensor in system x, then in any other system say $\bar{x}$, the components of this extensor will be $\delta^r{}_s$, $L^r{}_s$,—the quantities $L^r{}_s$ being linear forms in the $\bar{x}'$'s. Thus, as we would anticipate, the intrinsic derivative in the case of a geometry of paths is merely a contraction of extensors.

In what follows, we shall assume that the intrinsic derivative of a first order absolute tensor is a tensor of the same type as the original and try to express the intrinsic derivative of Mth order as a contraction of extensors. This will facilitate the computation of intrinsic derivatives and place in evidence the fact that certain quantities already known constitute the components of an extensor.

2. *Notation.* We shall, so far as feasible, distinguish the coordinate systems employed x and $\bar{x}$ by means of indices. Indicial letters belonging to the first part of the alphabet will be correlated to system x while the letters i to w will denote system $\bar{x}$. Intrinsic differentiation and the concept extensor involve differentiation with respect to the parameter of a parameterized arc. We shall indicate such differentiation by means of primes and enclosed Greek indices. Thus

$$x'^a = x^{a'} = dx^a/dt, \quad x^{(\alpha)a} = x^{a(\alpha)} = d^\alpha x^a/dt^\alpha.$$

The partial derivatives $\partial x^{(\alpha)a}/\partial x^{(\rho)r}$, (here $x^{(\rho)r} = d^\rho x^r/dt^\rho$, $x^r = \bar{x}^r$) will be denoted by the abridged symbol $X^{\alpha a}{}_{\rho r}$. Similarly, $X^{\rho r}{}_{\alpha a}$ will denote the partial derivative of $x^{(\rho)r}$ with respect to $x^{(\alpha)a}$.

3. *Extensors.* The summation convention used in the theory of extensors may be stated as follows: Lower case Latin and Greek letters indicate summations from 1 to N, and from 0 to M, respectively; while capital indices do not generate sums. We shall frequently replace α with A to forestall summation, thus

$$\binom{M}{\alpha} U_a{}^{\cdot\,(M-\alpha)}$$

with α not summed would be written

$$\binom{M}{A} U_a{}^{\cdot\,(M-A)}.$$

The extensor transformation law is exemplified by

$$T^{\rho r}_{\sigma s \cdot t} = T^{\alpha a}_{\beta b \cdot c} X^{\rho r}_{\alpha a} X^{\beta b}_{\sigma s} X^c{}_t.$$

Such an extensor would be described by saying that it is excontravariant of order one, excovariant of order one, and covariant of order one.

As an illustration of the extensor transformation law with the range of Greek indices 0 to 1, let us seek to construct an extensor of the type $L^a{}_{\beta b}$, which will be invariant in functional form, by taking for the class of components $L^a{}_{1b}$ (they constitute a tensor the Kronecker delta $\delta^a{}_b$. Thus by fiat $L^a{}_{1b}=\delta^a{}_b$ in system x. From the extensor transformation law, we have

$$L^r{}_{1s}=L^a{}_{\beta b}X^r{}_a X^{\beta b}{}_{1s}=L^a{}_{1b}X^r{}_a X^b{}_s=\delta^r{}_s$$

$$L^r{}_{s0}=L^a{}_{\beta b}X^r{}_a X^{\beta b}{}_{0s}=L^a{}_{0b}X^r{}_a X^b{}_s+\delta^a{}_b X^r{}_a X^b{}_s{}'$$

$$=L^a{}_{0b}X^r{}_a X^b{}_s+X^r{}_b X^b{}_{st}x'^t.$$

From the linearity of the second term in x'^t, we conclude that the desired invariance in functional form can be attained by letting $L_{0b}{}^a$ be a linear form $L_{bc}{}^a x'^c$ in the x''s with the coefficients functions of x. Thus the affine connection of the geometry of paths may be said to arise from the Kronecker delta and the extensor transformation law.

The algebra of extensors is essentially the same as that for tensors except that there are $M+1$ different contractions of a second order extensor and the quotient laws are perhaps a little less stringent than those of tensor analysis. Further, the fact that $X^{\rho r}{}_{\alpha a}$ vanishes whenever α exceeds ρ allows considerable lattitude with regard to the range in certain summations. The contraction theorem in the special form in which we shall employ it is essentially that of tensor analysis. It may be illustrated adequately by the following statement: If $T^{\rho r}{}_{\sigma s \cdot t}$ is an extensor of the type indicated by its indices, then $T^{\rho r}{}_{\rho r t}$ is a covariant tensor. The special quotient laws that we need allow us to assert that T is an extensor of the type indicated by its free indices whenever the same may be asserted of

$$V^{\alpha a}\,T^{c\cdots}_{\alpha a\cdots} \quad \text{or} \quad U_{\alpha a}\,T^{\alpha a\cdots}_{c\cdots}\ .$$

Provided in the first case that $V^{\alpha a}$ may be taken to be $V^{a(\alpha)}$ with V^a arbitrary and in the second case that $U_{\alpha a}$ may be taken to be

$$\binom{M}{A}\ U_a{}^{\cdot(M-A)}$$

with U_a arbitrary.

4. *The extensive derivative of $T^c{}_{\alpha a}$.* Since the Mth order intrinsic derivative of a contravariant vector V^a is obviously linear in the $V^{a(\alpha)}$ there is no theoretical difficulty involved in establishing its structure. One needs but detach the coefficients of the $V^{a(\alpha)}$. By the quotient law the resulting quantities constitute the components of an extensor—contravariant of order one and excovariant of order one. These quantities will obviously consist of Kronecker deltas, Christoffel symbols

and their products and derivatives. In order to facilitate the construction of this extensor, we shall introduce presently a simple scheme for manufacturing a new extensor from one of the type $T^c_{\alpha a}$. This scheme is so constructed that it will yield the components of an extensor associated with the $M+1$st intrinsic derivative in terms of the extensor associated with the Mth intrinsic derivative.

Let V^a be any vector whose components are sufficiently differentiable functions of the curve parameter t. Further, let $V^{\alpha a}$ be the extensor $V^{a(\alpha)}$. Finally, let $T^c_{\alpha a}$ be any extensor of the type indicated by its indices. Obviously, $V^{\alpha a}T^c_{\alpha a}$ is a contravariant vector for arbitrary choice of V^a, and the same may be asserted of its intrinsic derivative $\delta(V^{\alpha a}T^c_{\alpha a})/\delta t$. This intrinsic derivative is linear in $V^{\alpha a}$ (the range of α is now 0 to $M+1$) and hence has the form of a contraction. Therefore, the coefficients of $V^{\alpha a}$ in the intrinsic derivative will constitute a new extensor involving the $T^c_{\alpha a}$, $T^c_{\alpha a}{}'$, and a Christoffel symbol. We shall call the new extensor the *extensive derivative* of the original—since the range of the Greek indices is extended by the process. Obviously, if the $T^c_{\alpha a}$ are so selected that $V^{\alpha a}\,T^c_{\alpha a}$ is the Mth intrinsic derivative of V^a, then the contraction of $V^{\alpha a}$ over the range 0 to $M+1$, with the extensive derivative of $T^c_{\alpha a}$ will yield the $M+1$st intrinsic derivative. We now turn to the explicit development of the extensive derivative.

Following the notation of the geometry of paths, we shall use a letter, L, instead of braces for the components of connection. Thus the intrinsic derivative of $V^{\alpha a}T^c_{\alpha a}$ may be written in the form

$$(V^{\alpha a}T^c_{\alpha a})' + V^{\alpha a}T^b_{\alpha a}L^c_b$$

or, since $V^{\alpha a}$ in the present instance denotes $V^{a(\alpha)}$,

$$V^{\alpha+1\cdot a}T^c_{\alpha a} + V^{\alpha a}(T^c_{\alpha a}{}' + T^b_{\alpha a}L^c_b).$$

We next replace α with $\alpha-1$ in the first term and accordingly change the range from 0 to M to 1 to $M+1$. We may now exhibit the multiplier of $V^{\alpha a}$ by setting aside the term involving $V^{M+1\cdot a}$ which occurs in the first product and the "zero" term of the second product. To provide for the special case $M=0$, we introduce the factor $(1-\delta^M_0)$, which vanishes when M is zero and otherwise has the value unity. Thus the intrinsic derivative in question assumes the form

$$V^{M+1\cdot a}T^c_{Ma} + (1-\delta^M_0)\sum_{\alpha=1}^{M} V^{\alpha a}(T^c_{\alpha-1\cdot a} + T^c_{\alpha a}{}' + T^b_{\alpha a}L_b{}^c)$$
$$+ V^a(T^c_{0a}{}' + T^b_{0a}L_b{}^c),$$

or if we define the meaningless symbol $T^c_{-1\cdot a}$ to be zero,

$$V^{M+1\cdot a}T^c_{Ma} + \sum_{\alpha=0}^{M} V^{\alpha a}(T^c_{\alpha-1\cdot a} + T^c_{\alpha a}{}' + T^b_{\alpha a}L^c_b).$$

By virtue of the quotient law, the coefficients of the V's are the components of an extensor. We shall denote this new extensor—the extensive derivative of $T^c_{\alpha a}$ by the symbol $DT|^c_{\alpha a}$. The range of α in the derived extensor is 0 to $M+1$, one greater than the original range. Specifically, the components of $DT|^c_{\alpha a}$ are

$$(1)\quad DT|^c_{M+1\cdot a}=T^c_{Ma};\quad DT|_{\alpha a}{}^c=T^c_{\alpha-1\cdot a}+T_{\alpha a}{}^{c'}+T_{\alpha a}{}^bL_b{}^c,\quad \alpha<M+1.$$

These equations should be compared with Kawaguchi's recursion formula (equations 13.6 page 105 of reference 1). Unless I have overlooked a point Kawaguchi does not consider the question of constructing an extensor from his quantities $T_h{}^k$. The fact that this can be done readily may be deduced from his work. This particular paper of Kawaguchi's is very long (152 pages) and presents a great number of very important contributions to the theory of extensors. In addition, it contains an almost complete bibliography of the subject.

Now that we have established a satisfactory structure for the higher order intrinsic derivatives of V^a, let us turn to the question of the systematic computation of the fundamental extensor. First of all in order to have an intrinsic derivative, we must have an extensor $\delta^c{}_a$, $L^c{}_a$ of the type $g^c_{\alpha a}$, $\alpha=0,1$. With this as a basis, let us start with $M=0$ and the tensor $\delta^c{}_a$. The successive extensive derivatives of $\delta^c{}_a$ are computed as follows:

The original extensor $T^c_{\alpha a}$ is the tensor $\delta^c{}_a(M=0)$, i. e. $T^c{}_{0a}=\delta^c{}_a$.

$$D\delta|^c{}_{1a}=\delta^c{}_a,\quad D\delta|^c{}_{0a}=0+0+T^b{}_{0a}L^c{}_b=L^c{}_a.$$

We now let $T^c_{\alpha a}$ denote the first derived extensor $D\delta|^c_{\alpha a}$ and M now has the value one. The $T^c_{\alpha a}$ are given by the equations $T^c{}_{1a}=\delta^c{}_a$, $T^c{}_{0a}=L^c{}_a$. This is the original extensor. The second derived extensor is the extensive derivative of $D\delta|^c_{\alpha a}$ thus $D^2\delta|^c{}_{2a}=\delta^c{}_a$;

$$D^2\delta|^c{}_{1a}=DT|^c{}_{1a}=T^c{}_{0a}+T^c{}_{1a}{}'+T^b{}_{1a}L^c{}_b=L^c{}_a+0+L^c{}_a=2L^c{}_a;$$

$$D^2\delta|_{0a}{}^c=DT|_{0a}{}^c=0+T_{0a}{}^{c'}+T^b{}_{0a}L^c{}_b=L^b{}_a{}'+L^b{}_aL^c{}_b.$$

Likewise in order to compute the third extensive derivative, we let $T^c_{\alpha a}$ denote $D^2\delta|^c_{\alpha a}$. Thus assigning α values in decreasing order 2,1,0, $T^c_{\alpha a}$ now represents the extensor $\delta^c{}_a$, $2L^c{}_a$, $L^c{}_a{}'+L^b{}_aL^c{}_b$. By applying extensive differentiation to this last choice for $T^c_{\alpha a}$, we obtain for $D^2\delta|^c_{\alpha a}$ the extensor:

$$\delta^c{}_a,\ 3L^c{}_a,\ 3(L^c{}_a{}'+L^b{}_aL^c{}_b),\ (L^c{}_a{}'+L^b{}_aL_b{}^c)'+(L^b{}_a+L^d{}_a{}'L^b{}_d)L^c{}_b.$$

This suggests that $D^M\delta|^c_{\alpha a}$ has the form

$$\binom{M}{A}I^c_{Aa}$$

with the intermediate quantities I satisfying Kawaguchi's recursion formula $I^c_{\alpha-1\cdot a}=I^c_{\alpha a}{}'+I^b_{\alpha a}L^c_b$, $I^c_{Ma}=\delta^c_a$. To show that this is indeed true one needs but assume that it is valid for the case M and then compute $DT|^c_{\alpha a}$ by formula (1).

Thus in order to write down directly an intrinsic derivative of given order one may start with the Kronecker delta and apply repeatedly the process of differentiation plus contraction with L^c_b. The resulting quantities are the intermediate quantities $I^c_{\alpha a}$ (not the components of an extensor). Multiplication by the appropriate binomial coefficients gives the basic extensor—the Mth extensive derivative of the Kronecker delta—which we shall denote by $L^c_{\alpha a}$. Finally, the $L^c_{\alpha a}$ are multiplied by $V^{\alpha a}$ and summed. The result is the Mth order intrinsic derivative.

5. *The extensive derivative of* $T_c^{\alpha a}$. Evidently, we can develop a counterpart for the preceding theory by examining the intrinsic derivative of the vector $U_{\alpha a}T_c^{\alpha a}$, $U_{\alpha a}$ is the extensor

$$\binom{M}{A} U_a{}^{\cdot(M-A)}.$$

For the present we shall regard $T_c^{\alpha a}$ as an arbitrary extensor of the type indicated by its indices. Later we shall assume that it is such that $U_{\alpha a}T_c^{\alpha a}$ is the Mth order intrinsic derivative of the vector U_a.

Expanding the intrinsic derivative in question, we obtain

$$\delta(U_{\alpha a}T_c^{\alpha a})/\delta t=U_{\alpha a}{}'T_c^{\alpha a}+U_{\alpha a}T_c^{\alpha a\prime}-U_{\alpha a}T_b^{\alpha a}L_c^b.$$

We next express $U_{\alpha a}{}'$ and $U_{\alpha a}$ in terms of $U^*_{\alpha a}$—the corresponding quantities for the next higher value of M, $M+1$. Evidently,

$$\begin{aligned} U_{\alpha a}{}' &= \binom{M}{A} U_a{}^{\cdot(M+1-A)} \\ &= \binom{M}{A}\binom{M+1}{A}^{-1}\binom{M+1}{A} U_a{}^{\cdot(M+1-A)} \\ &= \frac{M+1-A}{M+1}\, U^*_{Aa} \end{aligned}$$

while

$$\begin{aligned} U_{\alpha a} &= \binom{M}{A} U_a{}^{\cdot(M-A)} \\ &= \binom{M}{A}\binom{M+1}{A+1}^{-1}\binom{M+1}{A+1} U_a{}^{\cdot(M+1-A-1)} \\ &= \frac{A+1}{M+1}\, U^*_{A+1\cdot a}\,. \end{aligned}$$

Substituting these expressions into the first equation, we get

$$\delta(U_{\alpha a}T_c^{\alpha a})/\delta t$$

$$=\sum_{\alpha=0}^{M}\left[\frac{M+1-\alpha}{M+1}U^*_{\alpha a}T_c^{\alpha a}+\frac{\alpha+1}{M+1}U^*_{\alpha+1\cdot a}(T_c^{\alpha a'}-T_b^{\alpha a}L_c^b)\right]$$

Setting aside the zero element of the first term and replacing α with $\alpha-1$ in the second term we have

$$\delta(U_{\alpha a}T_c^{\alpha a})/\delta t=U_{0a}T_c^{0a}$$

$$+\sum_{\alpha=1}^{M+1}U^*_{\alpha a}\left[\frac{M+1-\alpha}{M+1}T_c^{\alpha a}+\frac{\alpha}{M+1}(T_c^{\alpha-1\cdot a'}-T_b^{\alpha-1\cdot a}L_c^b\right].$$

Applying the quotient law, we are led to the conclusion that the quantities $D_1T|_c^{\alpha a}$ defined by

(2) $$D_1T|_c^{0a}=T_c^{0a};$$

$$D_1T|_c^{\alpha a}=\frac{M+1-A}{M+1}T_c^{Aa}+\frac{A}{M+1}(T_c^{A-1\cdot a'}-T_b^{A-1\cdot a}L_c^b).$$

constitute the components of an extensor of range $M+1$. We shall call the quantites $D_1T|_c^{\alpha a}$ the extensive derivative of the extensor $T_c^{\alpha a}$, or if it is necessary to distinguish it from the preceding extensive derivative, the *lower extensive derivative.*

Since the Kronecker delta δ_c^a may be regarded as a special extensor $(M=0)$ of the type $T_c^{\alpha a}$, we may compute its extensive derivative and assert that the result is an extensor of the type $T_c^{\alpha a}$ with $M=1$. Obviously, this process may be repeated as often as desired. We shall call the resulting extensor the Kawaguchi extensor.* Turning to the details of the calculation, we take as our first choice for $T_c^{\alpha a}$ the Kronecker delta δ_c^a and set $M=0$. Computing the first lower extensive derivative, we get

$$D_1\delta|_c^{0a}=\delta^a{}_c;\quad D_1\delta|^{1a}{}_c=-L^a{}_c,\quad M=1.$$

To obtain the components of the second extensive derivative $D_2\delta|^{\alpha a}{}_c$, we let $T^{\alpha a}{}_c$ be $D_1\delta|^{\alpha a}{}_c$, thus $T^{0a}{}_c=\delta^a{}_c$; $T^{1a}{}_c=-L^a{}_c$; $M=1$. Computing we have

$$D_2\delta|_c^{0a}=\delta^a{}_c;\quad D_2\delta|^{1a}{}_c=\tfrac{1}{2}(-L^a{}_c)+\tfrac{1}{2}(-L^a{}_c)=-L_c^a$$

$$D_2\delta|^{2a}{}_c=-L^a{}_c{}'+L^a{}_bL^b{}_c.$$

*See reference 1, equation (13.12), page 108. Kawaguchi's notation for these quantities conceals their extensor character, however he was probably the first to encounter them. After reexamining his work, I have discovered to my surprise that although our purposes and conclusions are different there is some overlapping in content.

Continuing the process, we get for the third derivative,

$$D_3\delta\,|\,^{0a}{}_c=\delta^a{}_c;\quad D_3\delta\,|\,^{1a}{}_c=-2/3L^a{}_c+1/3(0-L^a{}_c)=-L^a{}_c;$$

$$D_3\delta\,|\,^{2a}{}_c=1/3(-L^a{}_c{}'+L^a{}_bL^b{}_c)+2/3(-L^a{}_c{}'+L^a{}_bL^B{}_c=-L^a{}_c{}'+L^a{}_bL^b{}_c.$$

$$D_3\delta\,|\,^{3a}{}_c=(-L^a{}_c{}'+L^a{}_bL^b{}_c)'-(-L^a{}_b{}'+L^a{}_dL^d{}_b)L^b{}_c.$$

Thus the general rule for computing the higher order extensive derivatives of the Kronecker delta in case the subscribt is to retain its tensorial character is as follows. To obtain any class of components after the class in which α is zero, differentiate the preceding class and then subtract the contraction of this class with $L_c{}^b$ with b the dummy index.

To compute the Mth order intrinsic derivative of U_a directly, i.e., without first computing the preceding derivatives one may proceed in this manner: Start with the Kronecker delta and write out the Mth order lower extensive derivative by the rule just stated leaving sufficient room between the classes for subsequent contraction by the extensor $U_{\alpha a}$. In the present case binomial coefficients are incorporated in the $U_{\alpha a}$ while in the preceding case they are incorporated in the extensive derivative of the Kronecker delta.

BIBLIOGRAPHY

1. A. Kawaguchi, *Die Differentialgeometrie hoherer Ordnung I. Erweietrte Koordinatentransformationen und Extensoren*, Journal of the Faculty of Science, Hokkaiddo, Imperial University series I, vol. IX, nos. 1-2, 1940.

2. H. V. Craig, *On tensors relative to the extended point transformation*, Amer. J. Math. vol. 59, (1937), pp. 764-774.

3. H. V. Craig, *Vector and tensor analysis*, New York, 1943.

COLLEGIATE ARTICLES

Original Papers Whose Reading Does Not Presuppose Graduate Training

Equations Invariant Under Root Powering

by E. J. FINAN and V. V. MCRAE

Introduction. One of the operations sometimes performed in the study of algebraic equations is the formation of an equation whose roots are some positive integral power of those of a given equation.

For example, the Graeffe method* of solving equations depends upon such a process. Obviously, there are some equations such as $(x-1)^n=0$ which are invariant under such an operation since each root remains unchanged. Also $x^2+x+1=0$ is invariant under the root squaring process although each root is changed. The purpose of this paper is:

(1) To present a method of obtaining expressions for the roots of all polynomial equations of any degree which remain invariant when their roots are raised to any given positive integral power a.

(2) To obtain all irreducible equations with rational coefficients which remain invariant when their roots are raised to any given positive integral power a.

2. *Conditions for invariance.* Let

$$f(x)=x^k+c_1x^{k-1}+\cdots+c_{k-1}x+c_k=0 \quad (c_k\neq 0) \tag{1}$$

have the k distinct roots r_i. It is proposed to find the conditions on the r_i so that the equation whose roots are r_i^a where a is a given positive integer is also $f(x)=0$. Since most of the work is done on the roots rather than on the coefficients it is convenient to represent the roots of $f(x)=0$ as the unordered set $S:(r_1,r_2,\cdots r_k)$ and their ath powers as the set S^a. If $S=S^a$ the set S will be called *invariant under a-powering*, or *invariant*, and $f(x)=0$ will be said to *a-power into itself*. Thus one says that the set $(1,i,-i)$ is invariant under 7-powering but not invariant under 4-powering and that $(x-1)(x^2+1)=0$ 7-powers into itself.

Hence assuming $S=S^a$ and $a>1$, what are the conditions on the r_i? Beginning with r_1 form the sequence $r_1,r_1^a,r_1^{a^2},\cdots,r_1^{a^i}\cdots$. Since all of these quantities are in $S=S^a$ there is a first one $r_1^{a^g}$ for which

$$r_1^{a^g}=r_1^{a^f} \quad (f<g). \tag{2}$$

Then the set $S_1:(r_1,r_1^a,r_1^{a^2},\cdots,r_1^{a^{g-1}})$ is a subset of S consisting of g distinct elements. Since the elements of $S^a=S$ are distinct the elements of S_1^a are distinct, from which one gets

$$r_1^{a^g}=r_1. \tag{3}$$

From (3) and the formation of S_1 it follows one could have used any of the elements in S_1 as the first element and the others would be expressed as powers of it. This amounts merely to a cyclic

*Although the Graeffe method is not applicable to equations which are invariant under root-squaring, this fact is not mentioned in the articles on the subject examined by the authors. Of course, such equations may be solved by first applying a suitable transformation to the roots.

permutation of the elements. Hence any two such sets, with the same a, having one element in common are identical. From (3) and the condition $c_k \neq 0$, it follows that r_1 is a root of unity whose period divides a^g-1. If $g<k$ there is an element of r_2 of S not in S_1 from which a set S_2 may be formed having no elements in common with S_1. This process may be continued until S is exhausted. Finally if the period of r_1 is d and if $sd=a^g-1$ where g is minimal and positive, then $r_1^d=r_1^{sd}=r_1^{a^g-1}=1$ and $r_1^{a^g}=r_1$ from which it follows $S_1^a=S_1$.

These results are stated in

Theorem 1. *Let the roots $r_1, r_2, \cdots, r_k$ of $f(x)=0$ form the set S. A necessary and sufficient condition that $S=S^a$ where a is any positive integer is that S be a sum of nonoverlapping sets such as $S_1=S_1^a$ where r_1 is a root of unity whose period divides a^g-1 where g is minimal*

3. *Determination of all invariant sets.* To find all sets S of order k so that $S=S^a$ for a given a is essentially a matter of first finding all sets of the type S_1 for which $S_1=S_1^a$, and then forming all possible sets S of order k consisting of sums of sets of the type S_1. If an invariant set S of order k is not a sum of invariant sets of smaller order it will be called *primitive.* Otherwise S is *imprimitive.* The problem of finding all sets S_1 for a given a and g will now be considerd. Since by Theorem 1, the period of r_1 is a divisor of a^g-1, the elements of S_1 are among the (a^g-1)st roots of unity. These numbers form a cyclic group G of order a^g-1 under multiplication. The set S_1 is a cyclic subset or J set of order d and power a of this cyclic group.[1] Hence the problem of determining all primitive invariant sets S_1 of order g so that $S_1=S_1^a$ may be solved by determining all cyclic subsets of order g contained in the cyclic group of order a^g-1.

If R is that root of unity whose amplitude is $2\pi/(a^g-1)$ then the elements of G are R^i $(i=1,2,\cdots,a^g-1)$, the last being the identity element. It is more convenient here to represent G as an additive group whose elements are $1,2,\cdots,a^g-1$, with addition performed modulo a^g-1. A correspondence between the two representations of the group is

$$R^i \longleftrightarrow i (i=1,2,\cdots a^g-1). \tag{4}$$

The method of determining all sets S_1 for a given a and g will be made clear by the following illustration. Let it be required to find all primitive sets S_1 of order 6 which are invariant under squaring. Here $a=2$ and $g=6$ and by Theorem 1, r_1 is a 63rd root of unity. Hence it is required to find all cyclic subsets of order 6 and power 2 contained in the cyclic group G of order 63. A necessary[2] and sufficient

[1] For a definition and discussion of cyclic subsets see E. J. Finan, *Cyclic Subsets of a Group,* Duke Mathematical Journal, v. 12, No. 3, September, 1945, pp. 509-513.

[2] *Ibid.,* Theorem 1, p. 510.

condition that G contain a cyclic subset of order 6 and power 2 is that G contain an element r_1 of period e such that 2 belongs to 6 modulo e. An element of G is of period 63, 21, 9, 7, 3, or 1. By trial, 2 belongs to 6 modulo 63, modulo 21 and modulo 9. Since $\phi(63)=36$ and since all the elements of a cyclic subset are of the same period there are six sets S_1 each containing 6 elements of period 63. Similarly there are two sets S_1 each containing 6 elements of period 21 and one such set containing elements of period 9. There are no other primitive sets of order 6 which are invariant under squaring. In the following rectangular array each row contains a cyclic subset of G. The first 6 rows contain elements of period 63, the next two rows elements of period 21, and the last row elements of period 9. Each row is a primitive set invariant under squaring.

1	2	4	8	16	32
5	10	20	40	17	34
11	22	44	25	50	37
13	26	52	41	19	38
23	46	29	58	53	43
31	62	61	59	55	47
3	6	12	24	48	33
15	30	60	57	51	39
7	14	28	56	49	35

Since 2 belongs to 3 mod 7 and to 2 mod 3 and to 1 mod 1, the 9 elements of G not appearing in the above array fall into two sets of order 3 and one each of order 2 and 1, all of which are invariant under squaring.

To restate these results in terms of equations one could write that any equation of degree 6 whose roots are $\cos 2\pi/n+i\sin 2\pi/n$ where n runs through all of the entries in any one row of the above array, squares into itself. There are no other such equations of degree 6 except those which are products of equations of lower degree, each of which squares into itself.

It may be desirable to know the number of primitive sets of a given order n which are invariant under a-powering. This number[3] is exactly $f(a,a^n-1)/n=F(a,n)/n$ where

$$F(a,n)=a^n-\sum a^{n/p_i}+\sum a^{n/p_ip_j}-\cdots$$

and the p_i are the distinct prime factors of n. Thus for $a=2$ and $n=6$ there are $F(2,6)=2^6-2^3-2^2+2=54$ elements that fall into 9 sets of order six which are invariant under squaring. These are listed in the above illustration.

[3] For a derivation of this result and a discussion of the number theory functions f and F, see *Ibid.*, p. 512, and L. E. Dickson, *History of the Theory of Numbers*, v. 1, respectively.

4. *Irreducible invariant equations with rational coefficients.* Let it be required to find all rationally irreducible equations $f(x)=0$ of degree n with rational coefficients which a-power into themselves In view of the above discussion, in the complex field $f(x)$ may be written in the form $f=f_1f_2\cdots f_t$ of degrees $n_1,n_2,\cdots,n_t$ such that $\sum n_i=n$ and f_i divides $x^{a^{n_i}}-1$ for $i=1,2,\cdots t$. First assume that $t=1$ so that $f(x)$ is not a product of polynomials of lower degree each of which a-powers into itself. Then $f(x)$ divides $x^{a^n}-1$. As explained above the roots of $x^{a^n}-1=0$ may be represented as an additive group $G:(1,2,3,\cdots,a^n-1)$. The problem then is to find all cyclic subsets, of order n, of G whose elemnets correspond under the isomorphism (4) to the n numbers R^i which are the roots of an equation with rational coefficients.

If G contains a cyclic subset of order n then G must contain an element[4] r of period d so that a belongs to n modulo d. Since d divides a^n-1 the roots of $x^d-1=0$ occur among those of $x^{a^n-1}-1=0$. Under the isomorphism (4) the roots of $x^d-1=0$ correspond to those elements of G which are of the form $k(a^n-1)/d$ where k assumes the values $1,2,\cdots,d$. This last set of d numbers is a cyclic subgroup of G and its $\phi(d)$ generators are given by those values of k which are prime to d. However the equation of degree $\phi(d)$ whose roots are the numbers corresponding to $k(a^n-1)/d$ where k runs through the set of numbers less than d and prime to d, is the cyclotomic[5] equation $\psi_d(x)=0$ whose roots are the primitive dth roots of unity. It is convenient to say $\psi_d(x)=0$ is *determined by d.* This equation which has rational integral coefficients is known to be irreducible. Hence the roots of $f(x)=0$, which is rational, must be among the roots of the irreducible cyclotomic equation of degree $\phi(d)$ determined by d. The only possibility is for $f(x)$ to be this cyclotomic polynomial from which follows $\phi(d)=n$. Thus if $t=1$ and if $f(x)=0$ of degree n a-powers into itself, then a^n-1 has a divisor d such that $\phi(d)=n$.

Now if $t>1$ and f is a product of polynomials in the complex field each a-powering into itself, the corresponding cyclic subset has the structure $S:(r_{11},r_{12},\cdots,r_{1n_1};\ r_{21},r_{22},\cdots r_{2n_2};\ \cdots r_{k1},r_{k2},\cdots,r_{kn_k})$ where a belongs to n_i mod d_i while $n_1+n_2+\cdots+n_k=n$ and each of the subsets of order n_i is invariant under a-powering. As before the elements in the first subset correspond to numbers which are among the roots of the cyclotomic equation of degree $\phi(d_1)$ determined by d_1. Similarly the elements of the second subset of S correspond to numbers which are among the roots of the cyclotomic equation of degree $\phi(d_2)$ determined by d_2, etc. Now no two

[4] See footnote on page 42.

[5] For a discussion of cyclotomic equations see C. C. MacDuffee, *Introduction to Abstract Algebra*, pp. 105-108, or Bieberbach-Bauer, *Algebra*, Berlin, 1928, p. 228 *et seq.*

cyclotomic polynomials have a common factor unless they are identical. But since $f(x)$ is irreducible, it follows that the cyclotomic equation whose roots correspond under (4) to any subset of S is the same as the one whose roots correspond to any other subset of S or that the one cyclotomic equation contains all the roots that correspond to the elements in S. Hence $\phi(d_1)=n$. Also from the above discussion it follows that all of the subsets in S are of the same order and that $n=kn_i$. Thus all of the roots of $f(x)=0$ are among those of $x^{a^n-1}-1=0$.

Now suppose that a^n-1 has a divisor d such that $\phi(d)=n$. Then the cyclotomic equation of degree $\phi(d)$ determined by d a-powers into itself. To show this let the set $S_1:(1,r_2,r_3,\cdots,r_{\phi(d)})$ correspond to the roots of the cyclotomic equation determined by d. It is a cyclic subset of the additive group modulo d represented by $G_1:(1,2,3,\cdots,d)$. Since a is prime to d and since $S_1{}^a=(a,ar_2,ar_3,\cdots,ar_{\phi(d)})$ it follows[6] that $S_1{}^a=S_1$ or that the cyclotomic equation a-powers into itself. This concludes the proof of

Theorem 2. *A necessary and sufficient condition that there exist an irreducible rational equation $f(x)=0$ of degree n which is invariant under a-powering is that a^n-1 contain a factor d such that $\phi(d)=n$. Then $f(x)$ is the cyclotomic polynomial $\psi_d(x)$ determined by d.*

As an illustration suppose it is required to find all irreducible rational equations of degree 6 that are invariant under root-squaring. Here $n=6$, $a=2$, and $a^n-1=63$. Since 63 has two divisors such that $\phi(d)=6$, there are two such equations. They are

$$\psi_9(x)=(x^9-1)/(x^3-1)=x^6+x^3+1=0$$

and

$$\psi_7(x)=(x^7-1)/(x-1)=x^6+x^5+x^4+x^3+x^2+x+1=0.$$

The corresponding cyclic subsets are $S_1:(7,\ 14,\ 28,\ 56,\ 49,\ 35)$ and $S_2:(9,\ 18,\ 36;\ 27,\ 54,\ 45)$. The first consists of one set of order $n=6$ while the second illustrates the latter part of the above discussion with $k=2$, $n_1=n_2=3$ and $n=6$.

If $a=2$ and $n=7$ then $a^n-1=127$ which contains no factor d for which $\phi(d)=7$. Hence there are no irreducible rational equations of degree 7 that are invariant under root-squaring. Of course, there are reducible equations with rational coefficients of degree 7 such as $x^7-1=0$, which square into themselves.

[6] See, for example, C. C. MacDuffee, *Op. cit.*, p. 24.

On Graphical Approximations To the Mode

by Harold D. Larsen

Textbooks on statistical methods contain a variety of formulas for approximating the mode of a frequency distribution. Each of these formulas is of the form, $Mode=b+wF$, where b denotes the lower boun-

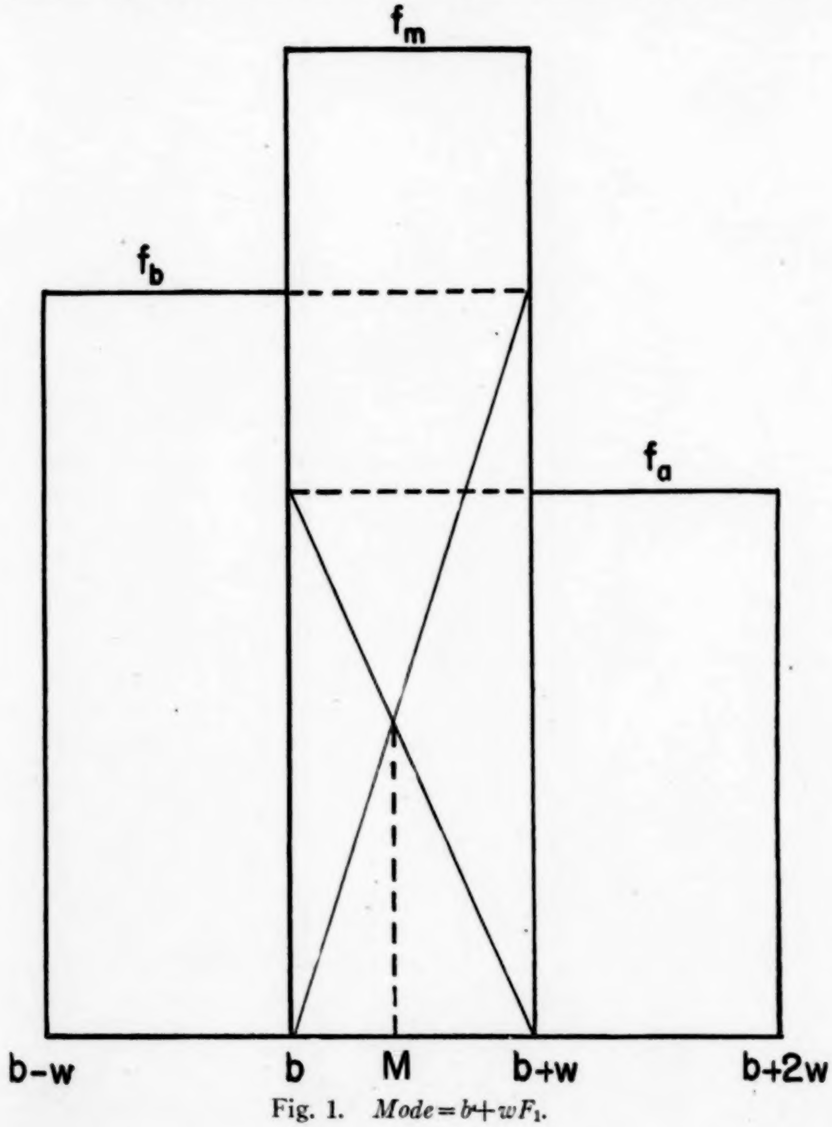


Fig. 1. $Mode=b+wF_1$.

dary of the modal class, w denotes the class width, and F is a proper fraction that varies from formula to formula. If f_m is the frequency of the modal class, f_a the frequency of the class just *above* the modal class, and f_b the frequency of the class just *below* the modal class, then the more common expressions for F are the following:*

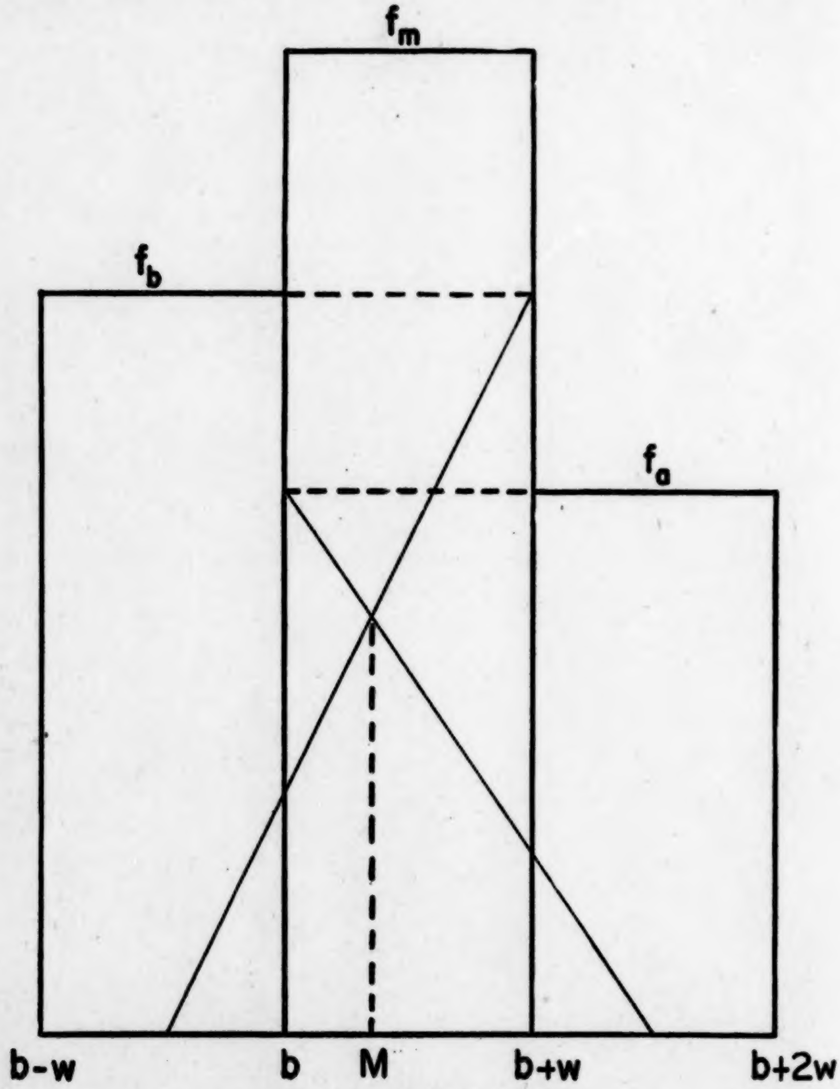


Fig. 2. $Mode = b + wF_2$.

*For proofs of these formulas see C. H. Richardson, *An Introduction to Statistical Analysis*, revised edition. New York, Harcourt, Brace and Company, 1944, pp. 80-86.

(1) $$F_1 = \frac{f_a}{f_a + f_b} \; ;$$

(2) $$F_2 = \frac{3f_a - f_b}{2(f_a + f_b)} \; ;$$

(3) $$F_3 = \frac{3f_a + f_m - f_b}{2(f_a + f_m + f_b)} \; ;$$

(4) $$F_4 = \frac{f_m - f_b}{2f_m - f_a - f_b} \; .$$

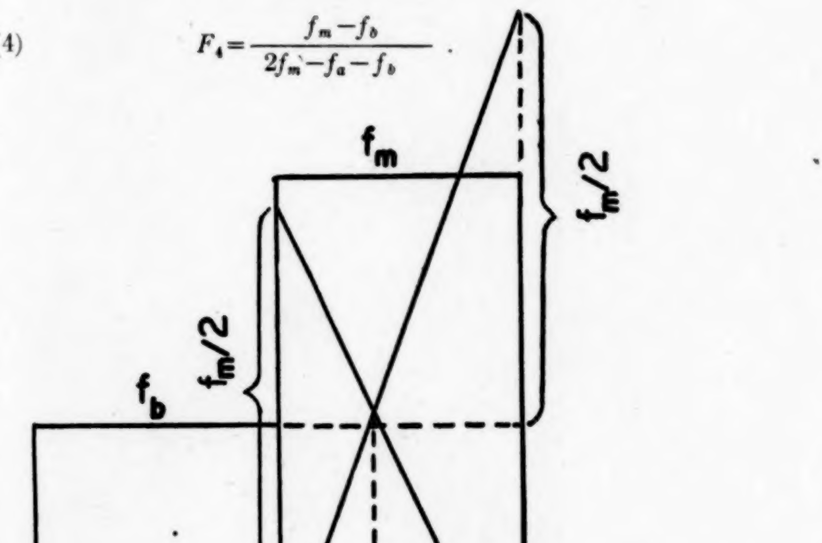


Fig. 3. $Mode = b + wF_3$.

Of these four expressions, F_1 is the most popular. It results if one assumes that the mode is at the centroid of two weights, f_a and f_b, suspended respectively at the upper and lower boundaries of the modal class. Contrariwise, if it is assumed that these same weights are suspended at the mid-points of their respective classes, F_2 results. The value for F_3 is obtained under the hypothesis that the mode is situated

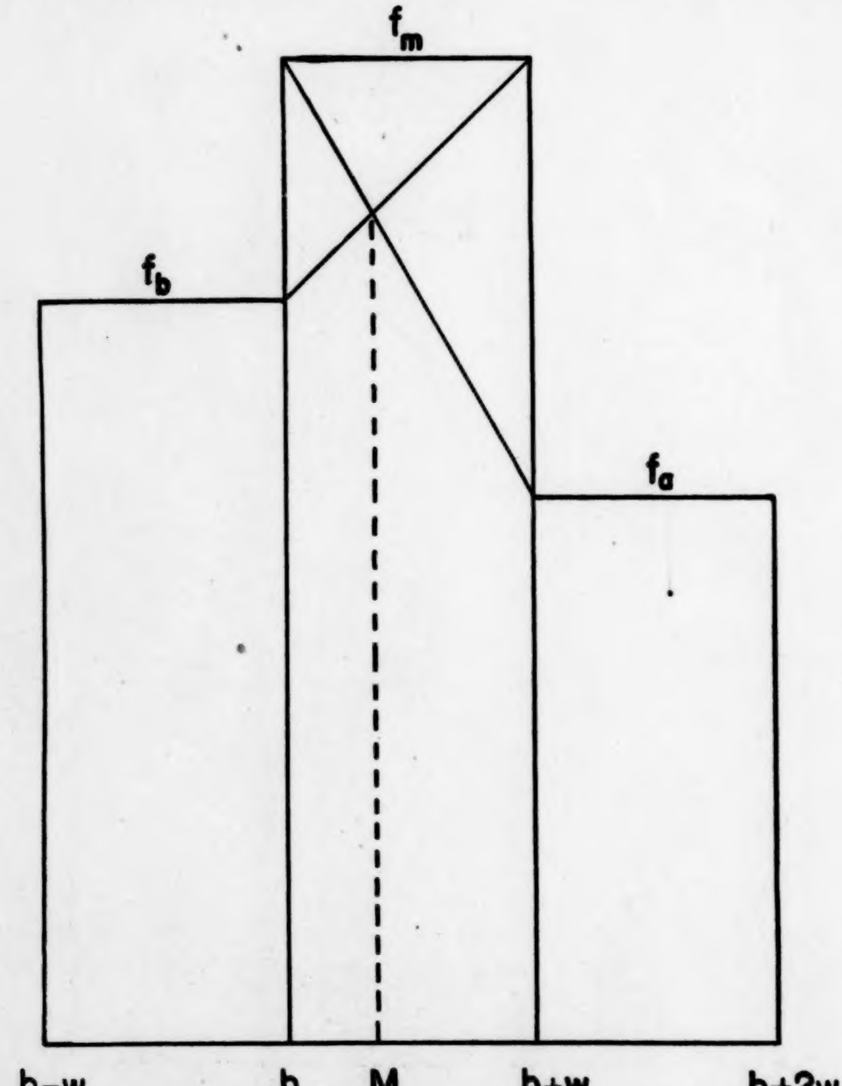


Fig. 4. *Mode* $= b + wF_4$.

at the centroid of three weights, f_a, f_m, and f_b, each suspended at the mid-point of its class. Finally, F_4 results from the assumption that the mode is equal to the abscissa of the maximum point on the parabola which passes through the three points, (x_a,f_a), (x_m,f_m), and (x_b,f_b), where x_i denotes the mid-point of the particular class.

Graphical methods for obtaining F_4 from the histogram are well known. However, it does not seem to be as well known that the values of F_1, F_2, and F_3 may also be obtained by simple graphical methods. These methods are described below. Since the proofs require at most simple analytic geometry, they are omitted here.

The following figures are self-explanatory. In each case, only three rectangles of the histogram have been drawn. The construction for F_4 is included for completeness. It is of interest to note that the easiest geometric construction corresponds to the most complicated algebraic determination of the mode.

The University of New Mexico.

A Number System Without a Zero-Symbol

by JAMES E. FOSTER

"Conceived in all probability as a symbol for an empty column on a counting board, the Indian *sunya* was destined to become the turning point in a development without which the progress of modern science, industry, or commerce is inconceivable."—Tobias Dantzig in ***Number, the Language of Science.***

Modern science, industry, and commerce have been possible because of an easily manipulated number system. This system is an historical consequence of the discovery of a zero-symbol. Does it follow, however, that an easily manipulated number system is impossible without the use of this symbol?

Consider a system (to be called the 0-less system) consisting of the numerals 1, 2, 3, 4, 5, 6, 7, 8, 9, T in which the numerals 1 through 9 correspond to their counterparts in the decimal system and in which T has the same value as 10. In this system, the following sequence would exist:

$\cdots 9, T, 11 \cdots\cdots 19, 1T, 21 \cdots\cdots\cdots 99, 9T, T1, T2 \cdots\cdots T9, TT, 111 \cdots\cdots 999, 99T, 9T1, 9T2 \cdots\cdots 9T9, 9TT, T11 \cdots\cdots T99, T9T, TT1 \cdots\cdots TT9, TTT, 1{,}111 \cdots\cdots .$

Sequences in the 0-less system, it will be noted, follow the same general pattern as those in the conventional decimal system which do not include numbers having 0 as a component part. Sequences involving numbers including T are consequences of this pattern in the absence of a zero-symbol. Thus, the successor of T is 11, that of $9T$ is $T1$, that of TTT is 1,111. In general, in the successor number T is replaced by 1 and its predecessor numeral is increased in value by 1. The nth power of T consists of $(n-1)$ 9's followed by T. ($T^3=99T$; $T^4=999T$; $T^6=99999T$). The multiplication of any number by T^n (where n is a positive integer) is written by subtracting 1 from that number and following the difference with T^n expressed as a number. ($15 \text{x} T=14T$; $15 \text{ x } 99T=1499T$; $TTT \text{ x } 9T=TT99T$).

The foregoing characteristics of the 0-less system determine the operations involved in addition, subtraction, multiplication, and division. Comparative operations under the decimal and the 0-less system follow for illustrative purposes:

Addition:

$$\begin{array}{r} 1{,}309 \\ 2{,}010 \\ \hline 3{,}319 \end{array} \qquad \begin{array}{r} 1{,}2T9 \\ 1{,}9TT \\ \hline 3{,}319 \end{array}$$

Subtraction:

$$\begin{array}{r} 7{,}568 \\ 3{,}459 \\ \hline 4{,}109 \end{array} \qquad \begin{array}{r} 7{,}568 \\ 3{,}459 \\ \hline 3{,}9T9 \end{array}$$

Multiplication:

$$\begin{array}{r} 105 \\ 246 \\ \hline 630 \\ 420 \\ 210 \\ \hline 25830 \end{array} \qquad \begin{array}{r} T5 \\ 246 \\ \hline 62T \\ 41T \\ 1TT \\ \hline 2582T \end{array}$$

Division:

$$\begin{array}{l} 246)25830(105 \\ 246 \\ \overline{}\;1230 \\ 1230 \\ \overline{} \end{array} \qquad \begin{array}{l} 246)2582T(T5 \\ 245T \\ \overline{}\;122T \\ 122T \\ \overline{} \end{array}$$

Since the multiplication of a number by T^n is indicated by subtracting 1 from the number and following the difference by T^n written as a number, the transformation of a fraction having T^n as a denomi-

nator to one in which the denominator is a greater power of T is a simple operation. ($17/9T = 16T/99T$) As a consequence, the manipulations involving decimal notation are possible in the 0-less system. It is true that the absence of a zero-symbol precludes the use of the decimal point in writing fractions having values less than 0.1. Such fractions, however, can be written in the 0-less system in the form $m(T^{-n})$. Thus 0.015 in the conventional system is equivalent to $15(T^{-3})$ in the 0-less system. For convenience in adding decimal fractions in the 0-less system, the power of T of the largest valued fraction can be indicated, and the numerators of the others can be written as they would in the conventional system with the 0's following the decimal point eliminated. As an illustration, consider the addition of identical quantities in the conventional and the 0-less systems.

.719	$719(T^{-3})$
.0235	235
.604	$5T4$
.00173	173
1.34823	1.34823

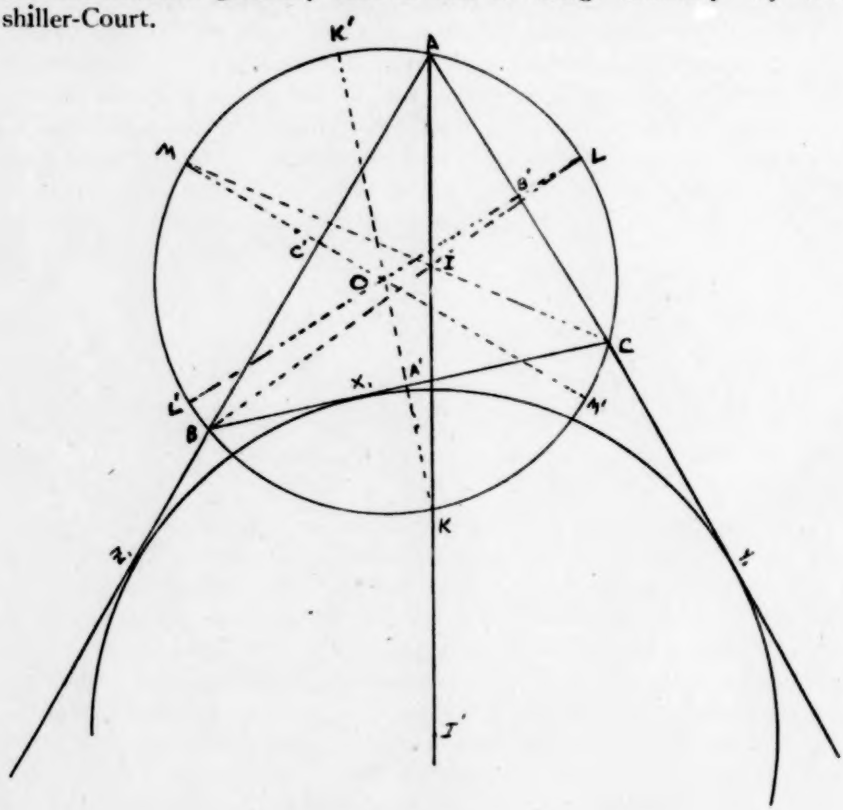
It will be noted that it is not necessary in the 0-less system to transpose decimal fractions to the highest negative power of T as a preliminary step to addition.

The foregoing manipulations indicate that the 0-less system has substantially the elasticity of the conventional decimal system, and as a consequence challenges the assertion that modern science, industry, or commerce would be inconceivable without the zero-symbol, even though its discovery happened to be an historical condition to their development. While facility in ordinary arithmetic manipulations may not be dependent on the symbol, it must nevertheless be realized that the development of pure mathematics would have been retarded without it, since the study of classes necessitates the identification of the 0-class. This paper, therefore, is not to be interpreted as an argument that the values of the zero-symbol to mathematics are wholly accidental, but as a discussion of its alleged essential character in an easily manipulated system of numbers.

A Note On Line Segments Connected With a Triangle and Its Related Circles

by F. A. LEWIS

The letters in the accompanying figure have the usual meanings and the notation generally follows that of *College Geometry** by Altshiller-Court.



From similar triangles $\dfrac{KC}{K'C} = \dfrac{r'}{p}$.

Similarly, $\dfrac{LA}{L'A} = \dfrac{r''}{p}$ and $\dfrac{MB}{M'B} = \dfrac{r'''}{p}$, so that

$$\frac{KC \cdot LA \cdot MB}{K'C \cdot L'A \cdot M'B} = \frac{r}{p}. \tag{1}$$

*See pages 71-75 for relations used in the simplifications.

By taking the sum and product of

$$\overline{KC}^2 = 2R \cdot A'K$$

$$\overline{LA}^2 = 2R \cdot B'L$$

$$\overline{MB}^2 = 2R \cdot C'M,$$

we may prove

(2) $$\overline{KC}^2 + \overline{LA}^2 + \overline{MB}^2 = 4R^2 - 2Rr$$

(3) $$KC \cdot LA \cdot MB = 2R^2 r.$$

Similarly,

(4) $$\overline{K'C}^2 + \overline{L'A}^2 + \overline{M'B}^2 = 8R^2 + 2Rr$$

(5) $$K'C \cdot L'A \cdot M'B = 2R^2 p$$

(6) $$K'A' + L'B' + M'C' = r' + r'' + r'''$$

(7) $$K'A' \cdot L'B' \cdot M'C' = \frac{Rp^2}{2}$$

(8) $$A'K + B'L + C'M = 2R - r$$

(9) $$A'K \cdot B'L \cdot C'M = \frac{Rr^2}{2}\ .$$

University of Alabama.

Current Papers and Books

Edited by
H, V. CRAIG

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

The purpose and policies of the first division of this department (Comments on Papers) derive directly from the major objective of the MATHEMATICS MAGAZINE which is to encourage research and the production of superior expository articles by providing the means for prompt publication.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited. Comments which express conclusions at variance with those of the paper under review should be submitted in duplicate. One copy wil be sent to the author of the original article for rebuttal. Comments which extend the results of a paper should be limited to two typewritten pages, longer articles should be submitted to the Research Department.

Communications intended for this department should be addressed to
H. V. CRAIG, *Department of Applied Mathematics*,
University of Texas, Austin 12, Texas.

Mathematics for Exterior Ballistics. By G. A. Bliss. John Wiley & Sons, New York, 1944. vii+128 pages. $2.00.

This book of six chapters and five tables gives an exposition of some of the mathematical devices used in calculating the trajectory of a projectile.

Chapter I indicates how artillery officers make use of maps and range tables. Chapter II sets up the differential equations for the standard trajectory. The Siacci theory, based on the assumption of low elevation firing, is explained in Chapter III. The next chapter takes up the integration of the standard equations either by the method of successive approximations, using Simpson's rule and interpolation formulas, or by use of the differential analyzer. Chapter V deals with corrections for following and cross winds, air density variation, etc. by use of functions of lines and adjoint systems of differential equations. The final chapter, Bombing from Airplanes, describes some of the mathematical problems involved and a linkage for solving the problem of hitting a target.

To a mathematician Chapters IV and V may well appear the most interesting: numerical integration and the application of functions of lines and adjoint differential equations. It will be interesting to compare Chapter VI with the corresponding information concerning actual practice in World War II, when it is released. The book under review serves well as an introduction to more extensive specialized treatises. It gives also a good example of the difficulties inherent in any application of mathematics to physical sciences.

Virginia Military Institute. W. E. BYRNE.

Plane and Spherical Trigonometry. By H. A. Simmons. John Wiley and Sons, Inc., New York, 1945. 387 pages, 116 pages of tables. $3.00.

This text begins with an exposition of rectangular Cartesian coordinates and then proceeds to the general definitions of the trigonometric functions. The text is characterized by its precision and care in statement of definitions. The author gives ample attention in the definitions to the functions of the quadrantal angles and to the failure of the definitions when division by zero arises. By use of the $(1,1,\sqrt{2})$—triangle and the $(1,\sqrt{3},2)$—triangle the author brings to the student in a clear manner the derivations of the functions of the positive and negative odd multiples of 45° and 30°, respectively. Chapter I closes with the definition of polar coordinates, the projection of a directed segment, and the definition of a vector.

Chapter II deals with the right-triangle definitions of the functions. At the end of this chapter forty-one problems are given. These problems are classified in groups as to subject matter. In this chapter, computation with approximate numbers is appropriately discussed. On page 47 a very useful table is given showing the corresponding accuracy in the measurements of lengths and angles.

Chapters III and IV deal with identities and reduction formulas, respectively. The author follows the desirable rule of expressing the angle to be reduced as an even or old multiple of 90° plus an acute angle. It would have been desirable in Chapter IV to have shown the use of the reduction formulas in finding the value of an angle given the value of the function. This last step is overlooked in many trigonometry texts.

"Radian and Mil Measure" are treated in Chapter V. This chapter devotes seven pages to certain approximate formulas on mil measures and their uses in naval and military science.

Chapter VI treats the graphs of the functions. Chapter VII covers the usual material on the addition formulas and their related identities. In Chapter VIII, devoted to logarithms,the author gives two pages to a discussion of the rules for finding the log sin — and log tan — for angles ranging from 3° to 0°. This topic is well placed. In Chapter IX the topics are arranged out of the ordinary. The first of the chapter is given over to the solution of right triangles by logarithms. This treatment is immediately followed by the four cases in the solution of the oblique triangle with logarithms. At the end of this chapter five basic problems on triangulation are given. Chapters X and XI treat inverse functions and equations, respectively. These two chapters are critically written as to meaning of the principal values and the general forms of the angles.

"Selected Material from Solid Geometry" is the topic of Chapter XII. Those portions of plane geometry needed are inserted throughout the chapter just prior to certain solid geometry theorems. This helps the student to be conscious of the requisite plane geometry in his solid geometry proofs. The whole chapter is built up toward the spherical geometry needed in Chapter XIII devoted to "Spherical Trigonometry." At the beginning of this chapter the definitions and theorems of solid geometry immediately needed in spherical trigonometry are given compactly. The development and discussion of the right spherical triangle formulas is given very fully. The chapter on spherical trigonometry is written with the view of being a full course. No special emphasis is placed on the haversine formulas in this chapter as is usual in spherical trigonometry built up primarily for the theory of navigation.

"Applications" is the title of the fourteenth and last chapter. This is a splendid introductory chapter on navigation. The material on the various types of sail-

ings is quite complete—just as complete as is usually found in a navigation text. At the end of this chapter ten pages are given on the celestial sphere and the use of the astronomic triangle. The author leads right up to the Marcq St. Hilaire method of navigation. He says nothing of lines of position or fixes.

The text contains appendices A, B, and C on ''Complex Numbers", ''The Slide Rule,' and ''Important Formulas." There is a set of answers to the odd-numbered exercises. Wiley's trigonometric tables are used with the text. In these tables a set of five-place natural values of the functions with the mil as the unit is included.

This is not just another trigonometry text. It is a job done by a keen scholar and by a teacher who senses class room procedure and the student's difficulties. The figures are well done and the format very pleasing. The reviewer recommnds an exhaustive examination of this text by anyone interested in adopting a trigonometry text.

Louisiana Polytechnic Institute. P. K. SMITH.

An Introduction to Mathematics for Teachers. By Lee Emerson Boyer. Henry Holt and Company, New York, 1945. xvii+478 pages. $3.25.

This would be an excellent textbook for a college class in methods of teaching mathematics in high school. It is well written,—giving a brief, historical background and a good review of each of the four courses discussed.

Part I. Arithmetic. The effort which man has made to establish our number heritage is described in an interesting and informative manner as is also the development of the signs of operations as we know them today. The historical development of improper fractions, irrational numbers, and negative numbers is well presented. The discovery of zero—claimed by Dantzig to be one of the greatest achievements of the human race—is called by Boyer the discovery of one of the oddest numbers of all as it stands by itself as a unique number indeed. In chapters III, IV, and V, Boyer has given the teachers of arithmetic a fine description of the fundamental operations of addition, subtraction, multiplication, division, raising to powers, and the extraction of the square and cube roots of integers. A teacher of arithmetic would enjoy the presentation of percentage as given in Chapter VI and the description of the slide rule and its use in Chapter VIII. Chapter IX on denominate numbers is very informative as to historical background and present day use. The treatment of the metric system is particularly good.

Part II. Algebra. Mr. Boyer has well described the relation of algebra to arithmetic and made it easy for readers to realize the power which operations with algebraic symbols adds to corresponding arithmetic power. The review of the fundamental operations, as applied to algebraic quantities, given in Chapters XI and XII is excellent. The application of algebra to the sciences is most interestingly presented in Chapters XIV and XV. The treatment of equations and graphs in this book is particularly good for the teacher of algebra.

Part III. Geometry. The beginning of geometry is cleverly depicted by Mr. Boyer in Chapter XX. The philosophy of reasoning to certain conclusions is told most graphically in Chapter XXI. The two kinds of geometry, informal and demonstrative, are well described in this book, which also gives a list of the most useful theorems of Euclidean geometry with associated figures and proofs.

Part IV. Trigonometry. Mr. Boyer gives a short but concise discussion of trigonometry, including a treatment of the properties of logarithms and the use of logarithms in the solution of triangles.

Texas Technological College. LIDA B. MAY.

HISTORY AND HUMANISM

Edited by
G. WALDO DUNNINGTON and A. W. RICHESON

STATEMENT OF POLICY

The history of mathematics forms the nucleus of the history of civilization; papers which stress this aspect of the subject will be welcome in this department. The value of mathematical history is a function of its accuracy. Hence the correction of widespread errors will be regarded as an important part of our historical work. Relationships between mathematics and other sciences must always stand in the foreground if we are to think of mathematical history in terms of the framework of general history. The humanistic phase of the influence of mathematics should not be neglected if we are to consider carefully the historical patterns of ideas and their long-range development over the centuries. This will help us to appreciate the great cultural heritage which the ages have bequeathed to us.

Papers which evaluate the sublime achievements of mathematical giants in the Golden Age will find a place in this department. Such research is bound to open up a vista of victories of the human mind in the past and make one optimistic as to the future. Furthermore, it will help us to understand the social, political and scientific milieu in which mathematical discovery and development have occurred. Occasionally the historian will uncover some mathematical fossil, which delights the general reader.

Humanistic implications in mathematical history are of prime importance, and to some extent the study of it may be looked on as a recreation. However, articles which are based on primary sources are more to be desired than a mere rehashing of already printed material. With this in mind, contributors to this department should always follow the principle that bibliography is a means rather than an end. Let bibliography (often an extravagant display of pedantry) be kept to a minimum. In the proper sense, it should help us to avoid duplication of work already done. Critical notes on items in a bibliography appreciably increase its value.

It is hoped that articles submitted to this department in the field of mathematical history will be varied. In other words, that some will follow the biographical method, that some will trace the history and development of ideas, and that others will trace mathematics in a given nation or region. Send papers to G. Waldo Dunnington, Northeastern State College, Natchitoches, Louisiana.

An Eleventh Lesson in the History of Mathematics

by G. A. MILLER

20. *Questionable historical statements.* In 1893 a meeting was held in Chicago, the proceedings of which were published in 1896 under the heading "Mathematical papers read at the international congress held in connection with the World's Columbian Exposition." It is now well known that in 1897, an important mathematical meeting was held in Zurich, Switzerland, the proceedings of which appeared in 1898 under the title *Erster Internationalen Mathematiker Kongress.* If the said meeting in Chicago was really an international congress as the title of its proceedings, which appeared as volume 1 of "Papers published by the American Mathematical Society", indicates the later meeting held in Zurich should clearly not now be regarded as the first international mathematical congress, as is now commonly done for good reasons as will be seen in what follows.

The custom became well established by the fact that four later meetings, which were similar to the noted one held in Zurich, Switzerland, were called in their publications the second, third, fourth and fifth International Mathematical Congress respectively. These are the meetings held in Paris (1900), Heidelberg (1904), Rome (1908) and Cambridge (1912). These later meetings were much more largely attended by mathematicians from different countries than the said meeting in Chicago, which was a comparatively small affair. According to the official register less than fifty people were in attendance at this meeting and only four of them were from outside the United States, while more than two hundred from sixteen different countries were in attendance at the said Zurich meeting. Moreover, the planning of this meeting was much more international than that for the Chicago meeting. The main fact to be emphasized in this connection is that in the history of mathematics it is frequently necessary to note changes of the meanings of the terms used. The said meeting in Chicago was held before the term international mathematical congress had received its now common meaning in the mathematical literature. This meeting, however, was of great importance in American mathematics and the somewhat exaggerated claims made in regard to its international character in some of the early American publications may be regarded as natural. The attending mathematicians however did not hesitate to ignore the fact that American mathematicians had so recently called the said Chicago meeting an international congress when they called the meeting held in Zurich in 1897 the first International Mathematical Congress.

Since the "History of Mathematics" by D. E. Smith (1860-1944) in two volumes (1923, 1925) is found in many of the school libraries and hence is read by many before they are in position to verify many of its statements it may be useful to consider here a few of the questionable statements contained therein. These considerations aim to inspire caution on the part of the reader even in the use of historical works of reference which have many attractive features and are widely used. On page 126 of volume 1 of this history by D. E. Smith there appears the following statement in regard to the well known Greek mathematician Heron. "He was able to solve the equation which we write in the form $ax^2+bx=c$, so that the general quadratic equation as we know it today was thus fully mastered by the Greek mathematicians."

Since the Greeks at that time knew nothing in regard to the theory of ordinary complex numbers, it is obvious that they could not have mastered the quadratic equation as we now understand this statement and hence it should be carefully considered by those who would like to determine the nature of the historical writings of D. E. Smith, since these are widely available appearing, for instance, in various parts of the fourteenth edition of the "Encyclopedia Britannica" of which he was mathematical editor and to which he contributed the articles on the history of mathematics, algebra, arithmetic, finger numerals and various other subjects.

In the third of these lessons, published in volume 15 (1941) of this *magazine*, we considered the history of the quadratic equation and some references relating to fundamental developments were given there. In particular, the graphic constructions of complex roots of such an equation were noted there. Such constructions were evidently far beyond the attainments of the ancient Greeks. *cf.* Miller, *Mathematical Gazette*, volume 12, 1925. It should be emphasized that the development of the number concept and the solution of the quadratic equations are closely related and such interrelations should be emphasized in the history of our subject since this history is greatly simplified by the grouping of closely related facts, and this, in turn, tends to develop a deeper insight into the subject itself as well as into the theory of mathematics.

To present further evidences of the fact that the said "History of Mathematics" by D. E. Smith contains some statements which are apt to convey incorrect notions relating to fundamental developments in mathematics we may note here that Chapter 10 of volume 1 begins with the following assertion: "Since this work is concerned primarily with the history of elementary mathematics, it would be quite justifiable to set its limit at the close of the 17th century." On the same page it is stated that "the algebra that is taught in the secondary schools and in the

freshman course in college was practically all in use before 1700. The symbolism has changed but little and although the elementary textbook is more extensive, it contains no mathematics that was not generally known before that date. The changes that have been made relate chiefly to methods of teaching and to applications of the subject."

These assertions relate to a long and very interesting period in the development of mathematics, especially in our own land. A history of elementary mathematics which would set its limit at the close of the 17th century could obviously contain comparatively little in regard to the developments of elementary mathematics in the United States as well as those resulting from such international movements as those connected with the International Commission on the Teaching of Mathematics, which was inaugurated during the Fourth International Congress of Mathematicians, held at Rome, Italy, in 1908. From the context it is clear that the quotations in question relate mainly to the first developments along certain lines, but the history of mathematics is much wider, and even with this restriction the noted assertions are questionable. In particular, the elements of determinants are now often included in elementary algebra and these were unknown in their present form in 1700.

On page 25 of volume 2 of the history under consideration it is stated that "no trace of the recognition of negative numbers, as distinct from simple subtrahends, has yet been found in the writings of the ancient Egyptians, Babylonians, Hindus, Chinese, or Greeks." This is a very fundamental historical statement in view of the great importance of the negative numbers in the development of mathematics and the fact that it is contrary to the views expressed by many other writers on the history of our subject. While the ancient Greeks calculated with such expressions as $a-b$ and Diophantus gave rules for the multiplication of added and subtracted quantities into each other, the abstract concept of negative numbers was not explicitly used by them.

The ancient Hindus, on the contrary, used frequently abstract negative numbers. Brahmagupta (born in 598) placed a dot over a number symbol to show that it was to be used as an actual negative number and various other Hindu writers used actual negative numbers. In fact, in their astronomical writings the ancient Babylonians used such rules of multiplication as are involved in the symbols

$$+-=-+=- \text{ and } --=+.$$

The recognition of debt as negative when a credit is regarded as positive is very ancient and the ancient Babylonians used equations in which one of the two members is a negative number, as was noted in the eighth

of these Lessons, published in volume 18 (1943) of this *Magazine*, where a large number of other facts in regard to negative numbers were noted.

The given excerpts from Smith's "History of Mathematics" do not aim to convey a complete idea of this history as a whole. A just estimate of its value would require a consideration of its strong points as well as its weak points. Unfortunately the beginner in the study of the history of mathematics frequently finds it difficult to secure an adequate knowledge of the latter in regard to writers who have become popular. D. E. Smith traveled extensively and in various countries and made many friends. He occupied various very influential positions and is one of the most widely known former writers on the history of elementary mathematics. Hence it is especially desirable for the student of this history to determine for himself to what extent some of his broad observations can be accepted as having been carefully considered.

Such statements as those noted above might profitably be discussed with a view to find supporting evidences which may have led to their formulation. For instance, it would be interesting to consider what contributions the ancient Greeks at the time of Heron had made towards the solution of the general quadratic equation as we know it today, which might have inspired the remark that they fully mastered it. In this connection it would be desirable to consider the article by J. Tropfke (1934) in which he gave forward steps in the solution of the quadratic equation during a period of about thirty-five hundred years. The main facts of this article are embodied in the third edition of volume 3 (1837) of his well known *Geschichte der Elementar-Mathematik*, where many fundamental developments relating to equations are treated and a large number of references are given.

The historical writings of Florian Cajori (1859-1930) are probably even more widely available to the American students of elementary mathematics than those of D. E. Smith and hence it is desirable that such students should be in position to have some means of judging the reliability of these writings. Many of them are based on the writings of the German mathematical historian Moritz Cantor (1829-1920), especially on his *Vorlesungen über Geschichte der Mathematik*, standard work on the subject until the Swedish mathematical historian G. Enestrom (1852-1923) directed attention to its numerous errors. He noted about 2000 corrections and improvements in the *Bibliotheca Mathematica*, volume 14, page 282 (1914). A considerable number of these apply also to the writings of Florian Cajori, especially to his "History of Mathematics" (1919), as can easily be verified.

On page 10 of this history the widely quoted myth that the ancient Egyptians constructed right angles by stretching a rope around three

pegs, separated by distances in the proportion of 3,4,5, is noted. This myth, which seems to be due to a mistranslation, is also stated as a historical fact by E. T. Bell in his "Development of Mathematics", page 64 (1940) although many writers had earlier called attention to the fact that it has no foundation. These writers include such well known historians as O. Neugebauer, *Geschichte der Antiken Mathematischen Wissenschaften*, volume 1, page 122 (1934).

As evidence of the thoughtless character of some of the important historical assertions of Florian Cajori, we note here that on page 93 of his history to which we referred there appears the following statement in regard to Hindu mathematics. "They advanced beyond Diophantus in observing that a quadratic has always two roots". While it is true that the existence of two roots of certain quadratic equations was recognized by Bhaskara, who was born in 1114, the Hindus obviously could not know that a quadratic has *always* two roots. Even the negative roots were usually not considered by them and they knew nothing about complex roots. Hence the noted quotation from Cajori's history is very misleading as regards a fundamental question in the history of elementary mathematics.

We shall consider here one more questionable statement from the same history in view of its surprising features. On page 149 there appears the assertion, "It is one of the greatest curiosities of the history of science that Napier constructed logarithma before exponents were used." It is well known that exponents appear, at least implicitly, in geometric series and that Archimedes already observed, in effect, that $a^m a^n = a^{m+n}$. Moreover, N. Chuquet used in his *Triparty* (1484) both negative and zero exponents with their modern meanings. This work does not seem to have had much influence at the time it was written. Many other writers used exponents before the time of John Napier (1550-1617). It is true, however, that neither John Napier nor J. Bürgi (1552-1632), who are commonly regarded as the founders of logarithms, seems to have had any clear conception of what we now generally call the base of a system of logarithm.

The questionable character of the quotation under consideration becomes more evident if it is noted that it is impossible to determine now when exponents were first used. In multiplication the ancient Egyptians employed successive doubling, which is equivalent to multiplying by different powers of 2, and on page 201 of vol. 1 of his *Vorlesungen uber Geschichte der Antiken Mathematischen Wissenschaften* (1934) O. Nengebauer gives a Babylonian table of powers of 9 from 9^2 to 9^6. Moreover, our common numerical notation represents different powers of 10 by different positions. The use of implied exponents seems to be very much older than the use of explicit exponents, but the use of the

latter should doubtless be said to have preceded the time when Napier constructed logarithms, and the quotations in question should inspire caution in the use of the writings of Florian Cajori, which are now largely out of date but are still widely read.

We referred above to a questionable statement in E. T. Bell's "Development of Mathematics" (1940). This volume is more recent than the writings of D. E. Smith and Florian Cajori to which we referred and hence one might at first be inclined to think that it containe fewer questionable assertions. This view does not seem to be supported by the evidences since E. T. Bell frequently contradicts himself therein. For instance, on page 112 it is asserted that the complete divorce of algebra and arithmetic was consummated only in the nineteenth century when the postulational method freed the symbols of algebra from any necessary arithmetical connotation," while on page 170 it is stated, on the contrary, that " we shall approach mathematical structure through the union affected in the nineteenth century between algebra and arithmetic."

In accord with the common use of the terms arithmetic and algebra there obviously never was a complete divorce of these subjects since the time when the subject of algebra was first developed, and hence the assertions that they were both divorced and united during the nineteenth century seems to have little meaning. At any rate, the assertion as it stands seems to be contradictory and such changes as are implied therein would require much more time than a single century. The main fact connected therewith is however that without a considerable amount of clarification such broad statements are confusing and of questionable value to the young reader, who is in danger of being misled thereby. It is usually better to start with no ideas about a subject than to start with false conceptions thereof.

On page 159 of this work by Bell it is stated that "the history of mathematics holds no greater surprise than the fact that complex numbers were understood, both synthetically and analytically, before negative numbers." It is clear that negative numbers are a special case of complex numbers and hence it would be impossible to understand the complex numbers without also understanding the negative numbers. It might be said that such questionable statements cannot do much harm since their contradictory nature can readily be observed. On the other hand, it will doubtless save time of many young readers if they know beforehand that they may expect to meet with some questionable remarks in case they are reading a book whose general style may attract their attention but which contains many such remarks.

Closely related to the remark about negative numbers considered in the preceding paragraph is the following quotation from page 158

of the volume in question. "The one glimmer of mathematical intelligence in the early history of negatives is the suggestion of Fibonacci that a negative sum of money may be interpreted as a loss." It should be noted that Fibonacci lived in the thirteenth century and is also known as Leonardo of Pisa. He wrote a well known book called *Liber Abbaci* of which the second edition (1228) is extant. From the facts that the ancient Babylonians used the terms *tab* and *mal* as we now use plus and minus to represent distances in opposite directions from a given straight line, and what was said above about negative numbers, it is clear that this quotation can properly be classed among the questionable remarks in the given work by E. T. Bell.

The student of mathematics usually proves for himself the various results which he accepts as true but in the history of mathematics such a course is often not feasible in view of the fact that the ground to be covered in the latter subject is much more extensive. The study of the early developments of our subject usually leads into various ancient languages and into literature which is not easily accessible. Hence the mathematical historian who desires to obtain a broad view of his subjects finds himself compelled to make use of various secondary sources of information. This accounts for the large number of errors which appear in historical writings and for the fact that the student of the general history of our subject finds it necessary to be extremely cautious in the selection of his authorities.

Extensive references to sources of information imply that the reader has access to a large library and is familiar with different languages. These references are therefore often of little value except to mature students. The best work for such students of the history of elementary mathematics is the *Geschichte der Elementar Mathematik* by J. Tropfke (1866-1939), which is devoted to the developments in mathematics up to, but not including calculus. This work appeared in two volumes (1902-3) but later it was revised and enlarged and was published in seven volumes, beginning to appear in 1921. A third edition began to appear in 1930 but J. Tropfke died before its completion. It contains a very large number of references and includes many recent developments, including some of those discovered by O. Nengebauer in regard to Babylonian and Egyptian mathematics. The successive editions enabled the author to correct a number of his earlier errors, as well as some made by others. Its main element of usefulness is, however, the large number of well selected references and the great care in its statements of conclusions.

The consideration of questionable historical statements is somewhat similar to the solution of exercises in a mathematical textbook. It serves to clarify ideas and to extend the developments of the text.

The answers to historical questions are however usually more intricate than those relating to the usual exercises in a mathematical textbook. The former frequently involves the consideration of long periods of mathematical developments and very gradual changes towards abstractions and generalizations which were slowly adopted by the public —closely related to the questionable statements are the questionable portraits of ancient mathematicians. D. E. Smith gives several such portraits in volume 1 of his "History of Mathematics" to which we referred above. In a review of this volume, published in *Isis*, volume 6, page 43 (1924), G. Sarton said "the inclusion of fanciful portraits seems to me a serious mistake."

Teaching of Mathematics

Edited by

JOSEPH SEIDLIN, L. J. ADAMS and C. N. SHUSTER

This department invites articles on methods of teaching, adaptation of subject matter and related topics. Some of the topics of current interest recently brought to our attention are: the problem of developing the individual student in present day large classes; the part played by examinations in training mathematicians; the use of the "grade curve" versus attainment standards; and how to meet the mathematical needs of freshman and sophomore engineers. Papers on such topics or on any other subject in which you as a teacher are interested, or questions which you would like to have our readers discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

A Method For the Point By Point Construction of Central Conics By Ruler and Compass

by FLOYD S. HARPER

Many interesting and ingenious methods for constructing conics are in the literature, but to my knowledge the simple method outlined below is not among them. The justification of the procedure furnishes an interesting example of the parametric representation of the central conics.

To locate points on the graphs of:

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1, \tag{1}$$

use is made of the familiar rectangle centered at the origin whose diagonals are the asymptotes,

$$Y = \pm \frac{b}{a} X, \tag{2}$$

of the hyperbola.

Solving equations (1) for x we have,

$$x = \pm \sqrt{a^2 \mp \left(\frac{ay}{b}\right)^2}. \tag{3}$$

If y is assigned the value Y,

$$\frac{ay}{b} = \frac{aY}{b} = X,$$

from equation (2), and the parametric equations of the conics are:

$$\begin{cases} x = \pm\sqrt{a^2 \mp X^2} \\ y = \dfrac{b}{a} X. \end{cases}$$

The problem of finding the x coordinates of points on the ellipse and the hyperbola corresponding to $y = Y$ is that of finding the values of $\sqrt{a^2 - X^2}$ and $\sqrt{a^2 + X^2}$ respectively. If coordinate paper is available this can easily be done with the compass.

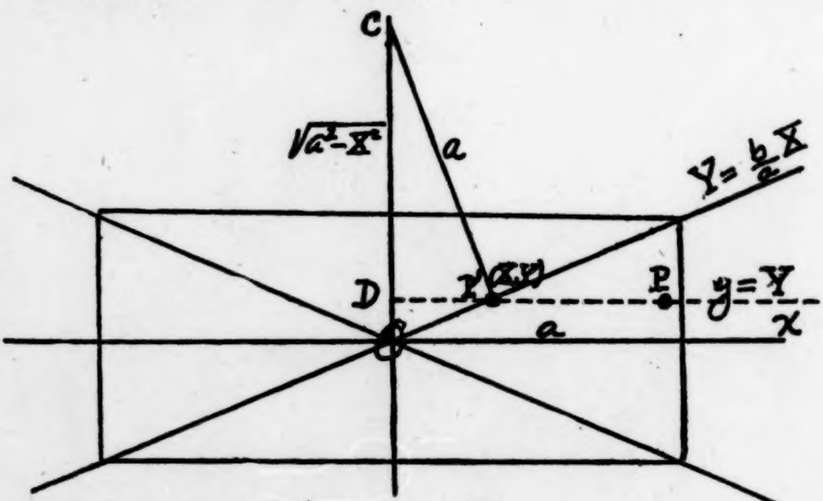


Figure I

In Figure I below use a as a radius with P' as center and strike an arc cutting Oy in C, then $DC=x$, which measured from D along the line $y=Y$ locates point P on the ellipse.

Similarly in Figure II below with X as radius and O as center strike an arc cutting Oy at C', then $AC'=x$, which measured from D along the line $y=Y$, locates point P on the hyperbola.

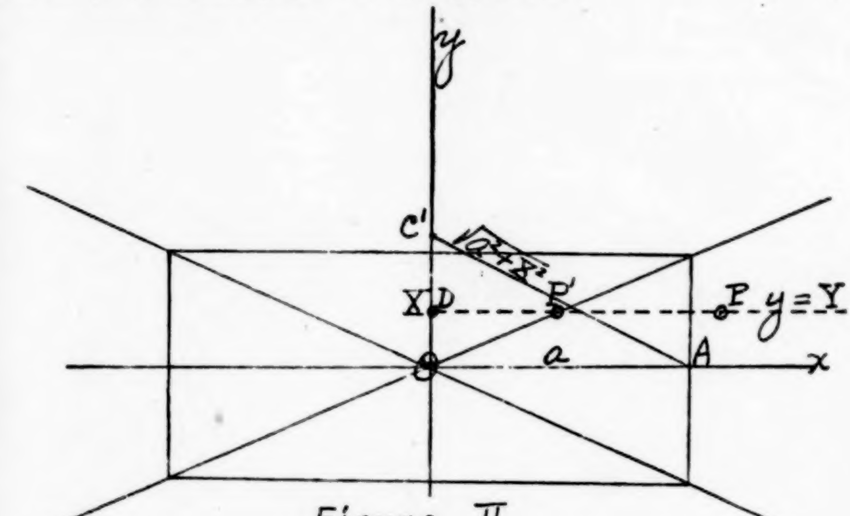


Figure II

The University of Nebraska.

Problems and Questions

Edited by
C. G. Jaeger and H. J. Hamilton

This department will submit to its readers, for solution, problems which seem to be new, and subject-matter questions of all sorts for readers to answer or discuss, questions that may arise in study, research or in extra-academic applications.

Problems and questions of interest, as such, are quite worth while but those that suggest lines of research or arise from some need are especially desired. All contributions will be published with the proposer's signature, unless we are instructed to the contrary.

Although no solutions or answers will normally be published with the offerings, they should be sent to the editors when known.

Send all proposals for this department to the Department of Mathematics, Pomona College, Claremont, California.

PROPOSALS

No. 1. Proposed by *V. Thebault*, Tennie, Sarthe, France.

In an orthocentric tetrahedron the spheres passing through three vertices and the feet of the corresponding altitudes have for radius the diameter of the first twelve point sphere and intersect by threes on the second twelve point spheres.

No. 2. Proposed by *E. P. Starke*, Rutgers University.

What restrictions must be placed on the four basic points of a coaxal pencil of concics, if all the conics of the pencil are to have (a) a common center, (b) a common axis of symmetry?

No. 3. Proposed by *Nev. R. Mind.*

If the medians of a triangle are proportional to the corresponding sides, the triangle is equilateral.

No. 4. Proposed by *Pedro A. Piza*, San Juan, Puerto Rico.

Let the integers a, b, c, with $c=a+1$, be the sides of a right triangle Show that

$$b^2c^2+a^4=a^2c^2+b^4$$

and that this value increased by 3 is a perfect square.

No. 5. Proposed by *Victor Thebault*, Tennie, Sarthe, France.

Using once each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, form a number which when increased by one million becomes a perfect square.

No. 6. Proposed by *Fred Fender*, South Orange, New Jersey.

A rocket carries half again its own weight in fuel. The fuel is used at a constant rate in 100 seconds and produces a jet of velocity 4900 meters per second. How high (to the nearest kilometer) will the rocket rise from rest if directed vertically. Neglect air friction, and assume the earth's gravitational field to be constant at 9.8 meters per sec^2.

No. 7. Proposed by *Pedro A. Piza*, San Juan, Puerto Rico.

Find squares of nine digits $a_1a_2a_3b_1b_2b_3c_1c_2c_3 = D^2$ where $a_1 \neq 0$, so that

$$a_1a_2a_3 = A^2, \quad b_1b_2b_3 = B^2, \quad c_1c_2c_3 = C^2$$

$$a_1a_2a_3 + b_1b_2b_3 + c_1c_2c_3 = F^2,$$

and

$$c_1c_2c_3b_1b_2b_3a_1a_2a_3 = E^2.$$

Mathematical Miscellany

Edited by
MARION E. STARK

Every reader will kindly look upon these words as a cordial invitation to send in items of general interest. Let us know (briefly) of *unusual* and successful programs put on by your Mathematics Club of new uses of mathematics, of famous problems solved, and so on. Brief letters concerning the MATHEMATICS MAGAZINE or concerning other "matters mathematical" will be welcome. Let us hear from you. Address: MARION E. STARK, Wellesley College, Wellesley 81, Mass.

D. H. Lehmer gives the value of e to 808 decimal places in a note on page 69 of *Mathematical Tables and other Aids to Computation*, (II, Number 14) for April, 1946. He is quoting from Peder Pederson, who in earlier calculations obtained e first to 404 decimals and then to 606 decimals.

"I know of no way in which we can serve our country better than by unswerving devotion to a crusade for the improvement of the teaching of college mathematics, and of no undertaking in which success would be more rewarding."

The Johns Hopkins University. F. D. MURNAGHAM.

(See also "The Teaching of College Mathematics", F. D. Murnaghan, in *The American Mathematical Monthly*, October, 1946.)

The Oriental abacus and the American calculating machine have been in competition with each other twice recently, with the abacus coming out ahead both times. In Tokyo, Kiyoshi Matsuzaki, of the Japanese Communications Ministry, finished five mathematical problems before Private Thomas N. Wood of Deering, Mo. The second contest (addition, subtraction and multiplication) took place in this country with P. T. So, a Chinese student from Canton now studying finance at Columbia University, defeating by eight seconds Miss Dorothy Boudreau from the WOR payroll department. This occurred on the "Better Half" broadcast of Station WOR. The abacus can be carried in a brief case, another advantage it possesses over the machine; but the machine lists the numbers added or multiplied and performs the desired operations, while with the abacus one cannot carry out division and must do multiplication mentally.

"Here is a brief exercise for you. Given sides b and c and angle A, Apprentice Seaman McSweeney desires to find a; and somewhat unwisely uses the Law of Cosines and logarithms according to the following ingenious method. He finds log b and doubles it, adds to that the double of log c, subtracts the sum of the logarithms of 2, of B, of c, and of cos A; halves the net; looks up the anti-logarithm, and to his instructor's great disgust produces the correct result. A/S McSweeney is annoyed at the grade he gets for all this work; and his instructor is curious to find under what conditions this 'method' works, assuming accuracy in the computation. Show that the necessary and sufficient condition for the validity of the McSweeney method is that the triangle be isosceles (the angle A being one of the equal angles)".

Dartmouth College. B. H. BROWN.

Harvard's new computation laboratory, first building of the proposed science center, was opened early in January of this year with a convention of two hundred or more mathematicians in attendance. The laboratory is the home of "Mark II", at present the largest calculating machine in the world. (*N. B.* An even more impressive "Mark III" is being planned at the Navy's request, to provide greater speed and flexibility than the present machine.) "Mark II", just now being completed, is all-electrical, very reliable and twelve times as fast as "Mark I", which was used for the Navy during the war.

In the discussions of scientists attending the convention mention was made of much active research at present on the subject of "remembering" or holding numbers in calculating machines until the numbers are needed later. The possibility of converting problems of economics

into pure mathematics and then solving them by calculating machines was discussed as a hope of the social sciences for the future.

At this convention Rear Admiral C. T. Joy, commanding officer of Dahlgren Proving Grounds, is quoted as saying "the new interest in mathematics might appear more likely to usher in an age of reason than the new interest in atomic energy."

Our Contributors

*James Alexander Shohat,** Professor of Mathematics at the University of Pennsylvania, died on October 8, 1944. His interesting article which appears in this issue was accepted for publication by the NATIONAL MATHEMATICS MAGAZINE, and now that the magazine is being revived we are happy to publish it.

Professor Shohat was born in 1886 in Brest-Litovsk, Russia and attended the University of Petrograd, where he received the degree of master ot pure mathematics in 1922. In the same year he married Nadiaschda Galli, a physicist, who later taught at Michigan, Mount Holyoke, Rockford College and Bryn Mawr. They came to the United States in 1923, after Professor Shohat had had several years experience teaching in Russian universities.

After several years in the United States at the University of Chicago and the University of Michigan, Professor Shohat spent a year doing research at the Institute Henri Poincare, Paris. Returning to this country in 1930, he joined the faculty of the University of Pennsylvania, where he remained until his death.

His main fields of research were orthogonal polynomials and approximation of functions. In addition to numerous research papers published in American and foreign mathematical journals, he published a monograph, "Theorie Generale des Polynomes Orthogonaux de Tchebichef" in the Memorial des Sciences Mathematiques series, and was co-author with J. D. Tamarkin of "The Problem of Moments," the first volume of the new series called *Mathematical Surveys*, published by the American Mathematical Society.

Professor Shohat was also interested in applied mathematics, and he acted as consultant to the David Taylor Model Basin, U. S. Navy, during the war.

He was a member of the American Mathematical Society, the Mathematical Association and the Institute for Mathematical Statistics, and was a fellow of the American Association for the Advancement of Science. During the last four years of his life Dr. Shohat was associate editor of the Bulletin of the American Mathematical Society.

His passing was indeed a loss to American mathematics.

Homer V. Craig, Professor of Applied Mathematics at the University of Texas, was born in Denver, Colorado in 1900. He attended the University of Colorado (A. B., 1924) and the University of Wisconsin (Ph.D., 1929), and then joined the faculty of the University of Texas.

His research interests are tensor analysis and related subjects. Professor Craig is also known as the author of a textbook on vector analysis. His hobbies are mountain climbing and the history of Colorado mining camps.

*A more comprehensive article by J. R. Kline, summarizing the work of Professor Shohat, was published in *Science* shortly after his death.

Edward J. Finan, Professor, Catholic University of America, was born in Nashport, Ohio, in 1900. He received the degrees of B. S. in E. E. (Dayton) 1922; M. A. (Ohio State) 1928; Ph.D. (Ohio State) 1930. An Assistant in Mathematics at Ohio State University 1926-30, he joined the faculty at Catholic University in 1930 as an instructor in Mathematics and has taught there without interruption since. Professor Finan is a member of the American Mathematical Society and of the Mathematical Association of America. His chief mathematical interest is in the field of elementary number theory.

Mr. V. V. McRae, the co-author of the article "Equations Invariant Under Root Powering," was a student of Professor Finan's at the Catholic University.

Harold D. Larsen was born in Warren, Pa., on December 7, 1904. He attended the Universities of Michigan and Wisconsin and received his doctorate at Wisconsin in 1936. From 1928 to 1935 he was an instructor at Wisconsin. Dr. Larsen came to the University of New Mexico in 1935 and was promoted by easy stages to full professorship. At the present time he is secretary of the Southwestern Section of the Mathematical Association. He is also a member of the American Mathematical Society, the Institute of Mathematical Statistics, American Association for the Advancement of Science, and the National Council of Teachers of Mathematics (New Mexico representative). Dr. Larsen is the editor of the *Pentagon*, official publication of Kappa Mu Epsilon. His hobbies are fishing, writing, and stamp collecting.

Fred A. Lewis, Professor of Mathematics, University of Alabama attended the University of Alabama and did graduate work at Johns Hopkins University where he received his doctorate in 1924. After teaching for a year at Texas A. and M., Professor Lewis came to the University of Alabama in 1920, and was appointed head of the department in 1945. He is a member of the Mathematical Association of America, American Mathematical Society, and Sigma Xi. His mathematical interests are algebra and groups, his hobby, philately.

George A. Miller, Emeritus Professor of Mathematics, University of Illinois, was born on a farm, July 31, 1863, in a community where both English and Pennsylvania German were used. The former was used at school, but the latter was more commonly used in the homes. At the age of seventeen he began to teach in the local public schools with a view to earning money to prepare for college and to pay his expenses at college. He graduated at Muhlenberg College, Allentown, Pa. in 1887, and soon thereafter secured the position of principal of the schools at Greeley, Kansas. A year later he was elected professor of mathematics at Eureka College, Eureka, Illinois, where he could devote more time to mathematics. After holding positions at Michigan, Cornell and Stanford Universities, he came in 1906 to the University of Illinois where he has remained ever since.

The recipient of numerous honors for his extensive researches in group theory, Professor Miller is also well known to the readers of the NATIONAL MATHEMATICS MAGAZINE for his series of articles on the History of Mathematics, the eleventh of which is published in the present issue.

The University of Illinois has published some of his Collected Works: volume I (1935) 475 pages; volume II (1938) 537 pages; volume III (1946) 499 pages. Additional volumes may be expected.

James E. Foster, Chief, informational service, Illinois Public Aid Commission; A. B., University of Illinois, 1923; Contributor to technical periodicals in engineering and related fields; author of original studies of propaganda as a social phenomenon lecturer by invitation to classes on public relations and social work at Northwestern University and George Williams College; member: Publicity Club (Chicago), American Public Welfare Association, Illinois Welfare Association; interest in mathematics is a hobby.

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This magazine is published by the Managing Editor at Los Angeles, California, over the periods January-February, March-April, May-June, September-October, and November-December.

The publisher assumes responsibility for supplying the magazines due those who have paid in advance for subscriptions to the *National Mathematics Magazine*. In the long run these subscribers will be an asset, but this year we need a goodly number of sponsor-subscribers and contributors to help "over the hump."

Send subscriptions to *Mathematics Magazine*, Glenn James, University of California, Los Angeles 24, California.

The following back numbers of the N. M. M. are available: Vols. XI, XII, XIV, XV, XVII complete, Vol. X except No. 1, Vol. XIII Nos. 6, 7, 8, Vol. XVI except No. 8, Vol. XVIII Nos. 1 and 2, Vol. XIX except Nos. 4 and 6, Vol. XX Nos. 1 and 2.

Due to insurmountable printing difficulties it has been necessary to start Volume XXI at this time instead of January-February as planned. All subscriptions and articles promised publication will be moved forward accordingly.—Ed.